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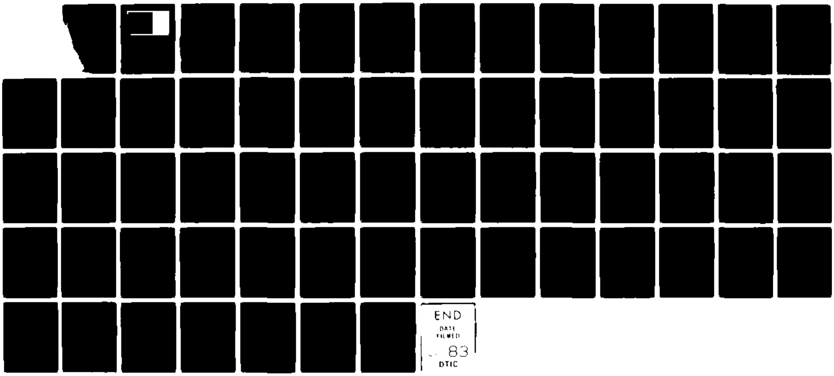

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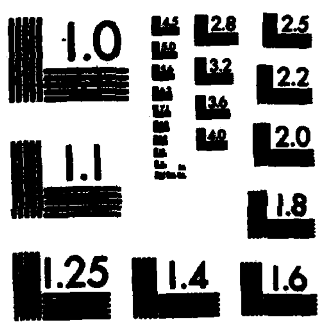

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LAX-WENDROFF METHODS FOR HYPERBOLIC HISTORY VALUE PROBLEMS

Peter Markowich and Michael Renardy

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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UNIVERSITY OF WISCONSIN-MADISON
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ABSTRACT

This paper is concerned with Lax-Wendroff methods for a class of hyperbolic history value problems. These problems have the feature that globally (in time) smooth solutions exist if the data are sufficiently small and that solutions develop singularities for large data. The author ~~We~~ prove (second order) convergence of the Lax-Wendroff method for smooth solutions and investigate numerically the dependence on the initial data. We demonstrate the occurrence of shock type singularities and compare the results to the quasilinear wave equation (without Volterra term).

AMS (MOS) Classifications: 35L67, 35L70, 45K05, 58C15, 65M10, 73F99

Key Words: Hyperbolic Volterra equations, Lax-Wendroff schemes, materials with memory, shocks, stability

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*Inst. f. Angew. und Num. Math., Technische Univ. Wien, A-1040 Wien, Austria.
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SIGNIFICANCE AND EXPLANATION

The motion of viscoelastic materials can be modelled by partial integrodifferential equations. For several model problems, recent investigations have been concerned with the question whether or not these equations allow the development of shocks. In this paper, we investigate this question numerically. Convergence proofs for a Lax-Wendroff type method are given. Our computed solutions confirm the predictions of previous papers. For small data, the solutions remain smooth, but for large data discontinuities develop.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

LAX-WENDROFF METHODS FOR HYPERBOLIC HISTORY VALUE PROBLEMS

Peter Markowich* and Michael Renardy**

1. INTRODUCTION

Viscoelastic materials are modelled by constitutive laws relating the stress to the history of the strain [6], [14], [15]. For most of the constitutive laws suggested by rheologists (for reviews see e.g. [1], [2], [16], [20]), the functional relationship has the form of a convolution integral. In many cases, the resulting integrodifferential equation of motion can be regarded as a perturbation of a hyperbolic equation [17].

This paper deals with the numerical analysis of a model equation of this form for one-dimensional viscoelastic solids, introduced by Dafermos and Nohel [3], [4]. The equation has the form

$$u_{tt} = \phi(u_x)_x - \int_0^t b(t-s)\psi(u_x(x,s))_x ds + f(x,t) , \tag{1.1}$$

where b is a positive, bounded, smooth, integrable kernel and ϕ, ψ are smooth functions satisfying

$$\phi(0) = \psi(0) = 0,\ \phi'(0) > 0,\ \psi'(0) > 0,\ \phi'(0) - \psi'(0)\int_0^\infty b(\tau)d\tau > 0 .$$

Particular interest in this problem arises for the following reason: It is well known that the quasilinear wave equation

$$u_{tt} = \phi(u_x)_x \tag{1.2}$$

need not have globally smooth solutions even if the initial data $u(t=0)$, $u_t(t=0)$ are smooth. Also, no damping occurs since the energy

$$E[u] = \int_{-\infty}^{\infty} u_t^2(x,t)dx + \int_{-\infty}^{\infty} \Phi(u_x(x,t))dx \tag{1.3}$$

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is a constant of the motion. Here $\Phi(u) = \int_0^u \phi(\sigma)d\sigma$. On the other hand, Dafermos and Nohel [3], [4] have shown that solutions to (1.1) on a finite interval (with Dirichlet or Neumann conditions on the boundary) remain smooth and decay to zero if the initial data and the forcing term are smooth and small in appropriate Sobolev norms.

The dissipative influence of the integral term thus acts against the formation of singularities. However, this effect is only strong enough for small initial data. The analysis of similar equations [8], [13], [24] has shown that, for bounded integral kernels (this is essential, cf. [18]) and large data, smoothness can be lost in finite time and shocks (i.e. discontinuities in u_x and u_t) can develop.

In this paper we discuss Lax-Wendroff type schemes for the numerical solution of (1.1). The equation is, for this purpose, transformed to a system by setting $u_x = v$, $u_t = w$.

In section 3 we show that - for appropriate integration methods approximating the Volterra term - these schemes converge of second order on any finite time interval on which a smooth solution exists. For the stability of this method, the nondegeneracy condition $\phi' \neq 0$ is essential, as is evident from recent work of Friedel and Osher [5].

In section 4 we show convergence uniformly up to $t = \infty$ for the case of spatially periodic small data and the special class of kernels

$$b(\sigma) = \sum_{\ell=1}^{M} K_\ell e^{-\lambda_\ell \sigma}, \quad K_\ell, \lambda_\ell > 0 . \tag{1.4}$$

Such kernels are commonly used in rheology. The proof is patterned after a new proof for the existence of globally smooth solutions, which we present in section 2. The essential ingredients for this proof are the stability of the trivial solution $u = 0$ and the quasilinear hyperbolic nature of the equations. If the λ_ℓ are very big, a stiffness problem arises in the numerical analysis, and we point out a simple modification of our scheme which avoids this.

Section 5 is concerned with numerical experiments. Computations demonstrate that singularities occurring for large data are indeed of shock-type (v and w have jumps). Since the Lax-Wendroff method is an artificial viscosity method the shocks are not sharp. A "shock layer" occurs whose width is proportional to the mesh size (see also Kreiss [12]).

Our computed solutions to (1.1) are compared to those for the corresponding quasilinear wave equation (1.2). This comparison shows that the dissipative mechanism of the Volterra term delays the time of the blow-up even if it is not strong enough to avoid the development of singularities.

It is well known that - for a scalar conservation law - the Lax-Wendroff method may converge to a nonphysical weak solution (which does not fulfill the entropy condition), see [5], [7]. For the problem (1.1), no theory of weak solutions and entropy conditions has been developed yet, and the main interest here is in computing globally smooth solutions or smooth solutions up to the blow-up time. For this purpose the second order Lax-Wendroff scheme is superior to first order methods, which do converge to the "right" solution for hyperbolic conservation laws.

However, our numerical results indicate that (1.1) has weak solutions with shock-type singularities.

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2. ANALYTICAL THEORY.

I- this section, we give a new proof for the global existence of smooth solutions in the case of small data. (The local existence can be shown using the ideas of Kato [9], [10], cf. [17]). The ideas used in this proof will serve as a guideline in our analysis of the numerical scheme in section 4. We assume that the kernel b has the form (1.4) and that we are dealing with spatially periodic solutions of (1.1): $u(x + 2\pi,t) = u(x,t)$. For simplicity, we also assume $\phi = \psi$. (For $\phi \neq \psi$, a similar, but more technical analysis is possible, see the remarks below). Under these conditions, the substitution $v = u_x$, $w = u_t$, $g_i = -u_t + \int_0^t e^{-\lambda_i(t-s)} \phi(u_x(x,s))_x ds$ yields the system

$$\dot{v} = w_x$$

$$\text{(2.1)} \qquad \dot{w} = \phi(v)_x - \sum_{i=1}^{M} K_i g_i - \Big(\sum_{i=1}^{M} K_i\Big) w + f$$

$$\dot{g}_j = -\lambda_j g_j - \lambda_j w + \sum_{i=1}^{M} K_i(g_i + w) - f$$

This is a perturbation of a hyperbolic system. In order to make it symmetric hyperbolic, we define: $\omega(y) = \int_0^y \sqrt{\phi'(y)}\, dy$, $\beta(y) = \omega'(\omega^{-1}(y))$ and set $\tilde{v} = \omega(v)$. Then (2.1) assumes the form

$$\dot{\tilde{v}} = \beta(\tilde{v}) w_x$$

$$\text{(2.2)} \qquad \dot{w} = \beta(\tilde{v})\tilde{v}_x - \sum_{i=1}^{M} K_i(g_i + w) + f$$

$$\dot{g}_j = -\lambda_j(g_j + w) + \sum_{i=1}^{M} K_i(g_i + w) - f$$

Clearly, β is positive in a neighborhood of 0. If $\phi \neq \psi$, (1.1) can still be transformed to a symmetric hyperbolic system after differentiating the equation [17]. An analysis similar to the following can then be applied.

The analysis will consist of three steps:

(i) Show that the linearization of (2.2) at $\tilde{v} = 0$, $w = 0$, $g_i = 0$ generates a semigroup of negative type. As a consequence, the inhomogeneous initial value problem has a unique bounded solution for $t \in [0,\infty)$, if the inhomogeneous term is bounded.

(ii) Show that the same property holds for the linearization of (2.2) at $\tilde{v} = \tilde{v}_0(x,t)$, $w = w_0(x,t)$, where $\tilde{v}_0, w_0$ are small in an appropriate norm.

(iii) Use a contraction argument for the nonlinear problem.

For brevity, let us write $\hat{z} = (\tilde{v}, w, g_1, \ldots g_n)$. The operator setting up the linearization of the right hand side of (2.2) at $\hat{z} = 0$ is denoted by A. This operator acts Fourier-componentwise, i.e. with $\hat{z} = \sum_{k\in\mathbb{Z}} \hat{z}_k e^{ikx}$ we have $A\hat{z} = \sum_{k\in\mathbb{Z}} A_k \hat{z}_k e^{ikx}$, where the matrix A_k is given by

$$A_k = \begin{pmatrix} 0 & ik\beta(0) & 0 & \cdots & \cdots & 0 \\ ik\beta(0) & -\Sigma K_i & -K_1 & \cdots & \cdots & -K_M \\ 0 & \Sigma K_i - \lambda_1 & K_1 - \lambda_1 & K_2 & \cdots & K_M \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \Sigma K_i - \lambda_M & K_1 & K_2 & \cdots & K_M - \lambda_M \end{pmatrix}$$

The characteristic equation of A_k is

$$\lambda^2 + k^2\beta^2(0) - k^2\beta^2(0) \sum_{i=1}^{M} \frac{K_i}{\lambda_i + \lambda} = 0 \tag{2.3}$$

<u>Lemma 2.1.</u>

All eigenvalues of A_k have negative real parts, except a double zero eigenvalue for $k = 0$.

<u>Proof.</u>

The function defined by the left hand side of (2.3) has simple poles at $\lambda = -\lambda_i$, and therefore has zeros between these poles. This accounts for $M - 1$ eigenvalues of A_k. A further eigenvalue lies between zero and $-\min_{i=1(1)M} \lambda_i$. For $\lambda > 0$, the left hand side of (2.3) is always positive. If λ is complex, the imaginary part of (2.3) yields

$$(2.4) \qquad 2\,\mathrm{Re}\,\lambda\,\mathrm{Im}\,\lambda - k^2\beta^2(0)\sum_{i=1}^{M}\mathrm{Im}\frac{K_i}{\lambda_i+\lambda} = 0$$

Since the sign of $\mathrm{Im}\frac{K_i}{\lambda_i+\lambda}$ is opposite to that of $\mathrm{Im}\,\lambda$, this is only possible if $\mathrm{Re}\,\lambda < 0$. For $k = 0$, it is easy to see that the eigenvalues of A_k are zero (two-fold) and $-\lambda_i$.

The presence of zero eigenvalues is inconvenient, but we can use a simple trick to get rid of them. It can be seen that the integrals $I_1 = \int_0^{2\pi} \omega^{-1}(\tilde{v})dx$ and $I_2 = \int_0^{2\pi} (\sum_{i=1}^{M} \frac{K_i}{\lambda_i}(g_i + w) - w)dx$ are invariants of the motion described by (2.2), provided that $\int_0^{2\pi} f dx$ is zero. We limit our attention to forces f satisfying this condition and to solutions for which the two integrals vanish initially. (Physically this means that the total force and the total momentum are zero). Such solutions also solve any modified equation, in which certain multiples of the two integrals are added to the right hand side of (2.2). Such a modification replacing the double zero eigenvalue by negative eigenvalues can be found. We shall refer to the so modified equation as (2.2') and to the modified linearized operator as A'.

As $k \to \infty$, two eigenvalues of A_k have the form $\pm ik\beta(0) + O(1)$, the others converge to distinct finite limits. Hence A_k is diagonalizable for large k. Let T_k be a transformation matrix such that $T_k^{-1}A_kT_k$ is diagonal. The eigenvectors of A_k setting up the columns of T_k also have limits as $k \to \infty$, and thus T_k can be normalized such that, as $k \to \infty$, it converges to a limit T_∞. T_∞ has the form

$$(2.5) \qquad T_\infty = \begin{pmatrix} 1 & -1 & 0 \cdots\cdots 0 \\ 1 & 1 & 0 \cdots\cdots 0 \\ 0 & 0 & \\ \vdots & \vdots & T_\infty^* \\ 0 & 0 & \end{pmatrix}$$

As a consequence, both T_k and T_k^{-1} stay bounded as $k \to \infty$. Therefore, by applying the transformation

$$\hat{z} = \sum_{k\in\mathbb{Z}} \hat{z}_k e^{ikx} = T\hat{y} = \sum_{k\in\mathbb{Z}} T_k\hat{y}_k e^{ikx},$$

the linearized operator in (2.2') assumes diagonal form (up to, maybe, a finite dimensional perturbation), and it is clear that the type of the semigroup generated by such an operator is determined by its eigenvalues. As an easy consequence, we obtain

Corollary 2.2.

For each $\hat{f} \in H^n([0,\infty) \times \mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{M+2})$ and each $\hat{z}_0 \in H^n(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{M+2})$, there exists a unique solution $\hat{z}$ of the equation $\dot{\hat{z}} = A'\hat{z} + \hat{f}$, which satisfies the initial condition $\hat{z}(x,0) = \hat{z}_0(x)$.

Here H^n denotes the usual Sobolev space, and n is arbitrary. This concludes step (i) of our analysis.

For abbreviation, let us now put

$$F(\tilde{v},w,g_1,\dots,g_M) = (\dot{\tilde{v}} - \beta(\tilde{v})w_x + I_1, \dot{w} - \beta(\tilde{v})\tilde{v}_x + \sum K_i(g_i + w) - I_2,$$

$$\{\dot{g}_j + \lambda_j g_j + \lambda_j w - \sum K_i(g_i + w) + I_2\}_{j=1}^M, \tilde{v}(t=0), w(t=0), \{g_j(t=0)\}_{j=1}^M)$$

For $n > 2$, F is a smooth mapping from $H^n([0,\infty) \times \mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{M+2})$ into $H^{n-1}([0,\infty) \times \mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{M+2}) \times H^{n-1}(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{M+2})$. Corollary 2.2 implies that the linearized mapping $DF(0)$ has an inverse. As a next step, we show that DF is invertible not only at 0, but in a neighborhood of 0 (step (ii)).

Lemma 2.3.

If $\hat{z}^0 \in H^n([0,\infty) \times \mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{M+2})$ $(n > 2)$ has sufficiently small norm, then $DF(\hat{z}^0)$ has an inverse which is bounded as a mapping from $H^n([0,\infty) \times \mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{M+2}) \times H^n(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{M+2})$ into $H^n([0,\infty) \times \mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{M+2})$.

Sketch of the Proof:

We show L^2-invertibility. Estimates for the derivatives are then easily obtained by successively differentiating the equation. The problem lies in the fact that the perturbations resulting from the terms $-\beta(\tilde{v})w_x$ and $-\beta(\tilde{v})\tilde{v}_x$ are not bounded relative to A', and hence we need a more refined argument than standard perturbation theory.

It is well known [10] that, for $\beta(\tilde{v}^0) \in H^2(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$, the operator $\tilde{B} : (\tilde{v},w) \to (\beta(\tilde{v}^0) - \beta(0))(w_x,\tilde{v}_x)$ is a bounded perturbation of a skew-adjoint operator B. Therefore, the linearization of (2.2') has the following structure

$$(2.6) \qquad \dot{\hat{z}} = A'\hat{z} + B\hat{z} + C\hat{z} + \hat{f}$$

where A' is as above, B is skew-adjoint and $C : L^2 \to L^2$ has small norm. Above, we indicated how to construct a transformation $T = (T_k)_{k\in\mathbb{Z}}$ such that A'_k is diagonalized, or, if it has a degenerate eigenvalue for finite k, transformed to Jordan form. Since all eigenvalues are negative, we can choose T such that $T^{-1}A'T$ is dissipative: $(\hat{z},T^{-1}A'T\hat{z}) < -\sigma(\hat{z},\hat{z})$ for some $\sigma > 0$. Moreover, as $k \to \infty$, we have $T_k = T_\infty + O\left(\frac{1}{k}\right)$ with T_∞ of the form (2.5). As a consequence $T^{-1}BT - T_\infty^{-1}BT_\infty$ is bounded (B is a first order differential operator, but $T_\infty - T$ is of order minus one). Moreover, it is easy to see that $T_\infty^{-1}BT_\infty$ is still skew-adjoint. It follows that $T^{-1}(A' + B + C)T$ is dissipative. This implies the lemma.

We have thus shown the invertibility of the linearization in a neighborhood of 0. At this point, generalized implicit function theorems (see e.g. [21]) would be applicable, however, the quasilinear nature of (2.2') admits a simpler argument. Namely, we can write $F(\hat{z})$ in the quasilinear form $F(\hat{z}) = L(\hat{z})\hat{z}$, where $L(\hat{z})\hat{z}' = DF(0)\hat{z}' + (\beta(0) - \beta(\tilde{v}))(w'_x,\tilde{v}'_x,0,\ldots,0)$. We wish to solve the inhomogeneous problem $F(\hat{z}) = \hat{f}$. Since we have just shown that $L(\hat{z})$ is invertible for small enough $\hat{z}$, we try the iteration $\hat{z}_m = L^{-1}(\hat{z}_{m-1})\hat{f}$. If the H^n-norms of $\hat{f}$ and the starting value $\hat{z}_1$ are small, the $\hat{z}_n$ stay in a small neighborhood of zero in H^n. Moreover, we have

$$L(\hat{z}_{m-1})\hat{z}_m = L(\hat{z}_m)\hat{z}_{m+1} \Longrightarrow L(\hat{z}_m)(\hat{z}_{m+1} - \hat{z}_m) = (L(\hat{z}_{m-1}) - L(\hat{z}_m))\hat{z}_m$$

whence we find the estimates

$$\|\hat{z}_{m+1} - \hat{z}_m\|_{n-1} < C\|\hat{z}_{m-1} - \hat{z}_m\|_{n-1}\|\hat{z}_m\|_n$$

If the $\|\hat{z}_m\|_n$ are small enough, then the sequence $\hat{z}_m$ converges in H^{n-1}, and it is immediate that the limit must be a solution. We thus find

<u>Theorem 2.4.</u>

If $f \in H^n([0,\infty) \times \mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$ and $\tilde{v}(0),w(0),g_1(0),\ldots,g_n(0) \in H^n(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$ $(n > 2)$ have sufficiently small norms, then (2.2') has a solution $(\tilde{v},w,g_1,\ldots,g_n) \in H^n([0,\infty) \times \mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{m+2}$ which assumes the prescribed initial values.

3. LAX-WENDROFF TYPE SCHEMES - LOCAL ANALYSIS

In this section we do not require the kernel b to have the special form (1.4). It is only assumed that b is sufficiently smooth and that b, b' are in $L^1(\mathbb{R}^+)$.

Equation (1.1) is transformed to a system by setting

$$v = u_x \, , \quad w = u_t \, . \tag{3.1}$$

This yields

$$\text{(3.2)} \qquad \begin{aligned} &\text{(a)} \quad v_t = w_x \\ &\text{(b)} \quad w_t = \phi(v)_x - b*\psi(v)_x + f(x,t) \end{aligned}$$

with the initial data

$$\text{(3.3)} \qquad \begin{aligned} &\text{(a)} \quad v(x,0) = u_0'(x), \qquad x \in \mathbb{R} \\ &\text{(b)} \quad w(x,0) = u_1(x), \qquad x \in \mathbb{R}\, . \end{aligned}$$

The star denotes the convolution

$$b*p(t) = \int_0^t b(t-s)p(s)ds \, . \tag{3.4}$$

We discretize (3.2), (3.3) using a rectangular mesh. The values of the approximations to v,w at the grid points are denoted by a lower spatial and an upper temporal index v_i^n, w_i^n. The following definition explains the details of the notation.

Definition 3.1.

Let $h > 0$, $k > 0$. For $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ we set $t_n = nk$, $x_i = ih$. For $\tilde{u} = (u_i^n)_{i\in\mathbb{Z}, n\in\mathbb{N}}$ we define the "spatial" difference quotients

$$\Delta^+ u_i^n = \frac{u_{i+1}^n - u_i^n}{h}$$

$$\Delta^- u_i^n = \frac{u_i^n - u_{i-1}^n}{h}$$

$$\Delta\, u_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{2h}$$

and the "temporal" difference quotient

$$\delta^+ u_i^n = \frac{u_i^{n+1} - u_i^n}{k} .$$

The idea of the Lax-Wendroff method is to approximate $v(x_i,t_{n+1})$, $w(x_i,t_{n+1})$ by the truncated Taylor series

$$\text{(3.5)} \qquad \begin{aligned} &\text{(a)} \quad v(x_i,t_{n+1}) \approx v(x_i,t_n) + kv_t(x_i,t_n) + \frac{k^2}{2} v_{tt}(x_i,t_n) \\ &\text{(b)} \quad w(x_i,t_{n+1}) \approx w(x_i,t_n) + kw_t(x_i,t_n) + \frac{k^2}{2} w_{tt}(x_i,t_n) \end{aligned}$$

and to substitute for the t-derivatives the corresponding expressions obtained from (3.2). The x-derivatives in these expressions are approximated by spatial difference quotients in a symmetric way. This yields

$$\text{(3.6)} \qquad \begin{aligned} &\text{(a)} \quad v_i^{n+1} = v_i^n + k\Delta w_i^n + \frac{k^2}{2} [\Delta^+\Delta^-\phi(v_i^n) - \Delta^+\Delta^-(b \overset{k}{*} \psi(v_i^n)) + \Delta f_i^n] \\ &\text{(b)} \quad w_i^{n+1} = w_i^n + k[\Delta\phi(v_i^n) - \Delta(b \overset{k}{*} \psi(v_i^n)) + f_i^n] \\ &\qquad + \frac{k^2}{2} [\Delta^+\left(\phi'\left(\tfrac{1}{2}(v_i^n + v_{i-1}^n)\right)\Delta^- w_i^n\right) - b(0)\Delta\psi(v_i^n) - \Delta(b' \overset{k}{*} \psi(v_i^n)) + (f_t)_i^n] \end{aligned}$$

for $n \geq 0$, $i \in \mathbb{Z}$, where we denoted

$$\text{(3.7)} \qquad f_i^n = f(x_i,t_n), \; (f_t)_i^n = f_t(x_i,t_n)$$

and $\overset{k}{*}$ is the discrete convolution operator

$$\text{(3.8)} \qquad b \overset{k}{*} p^n = k \sum_{j=0}^{n} \alpha_{jn} b(t_{n-j}) p^j$$

We require that the weights α_{jn} fulfill

$$\text{(3.9)} \qquad 0 \leq \alpha_{jn} \leq \alpha, \; \sum_{j=0}^{m} |\alpha_{j,m+1} - \alpha_{jm}| \leq C ,$$

where C is independent of m. For example, for the trapezoidal rule $\alpha_{0n} = \alpha_{nn} = \frac{1}{2}$,

$\alpha_{jn} = 1$ for $j = 1\ (1)\ n-1$ holds. The initial data are

$$\text{(3.10)} \quad \begin{aligned} &\text{(a)} \quad v_i^0 = u_0'(x_i)\,, \quad i \in \mathbb{Z} \\ &\text{(b)} \quad w_i^0 = u_1(x_i)\,, \quad i \in \mathbb{Z}\,. \end{aligned}$$

If $\overset{k}{*}$ is at least second order accurate for sufficiently smooth p and b, i.e. if

$$\text{(3.11)} \qquad |b \overset{k}{*} p^n - \int_0^{t_n} b(t_n - s)p(s)ds| = O(k^2)$$

holds, then the Lax-Wendroff method (3.6), (3.10) is second order accurate (as far as the local discretization error is concerned), provided that the data and solution are smooth enough.

Thus our scheme is second-order consistent. In order to show convergence, we have to prove stability [11]. Since the linearization is Lipschitz continuous with a Lipschitz constant of order $O(\frac{1}{h})$, it is sufficient to show stability for the linearization at the exact solution, it is, however, not enough to show stability for the linearization at zero, even if we are dealing with small solutions. Nevertheless, we analyze the stability at the trivial solution first, since this will also give us a guideline for dealing with the variable coefficient problem. We thus linearize (3.6) at $v_i^n \equiv w_i^n \equiv 0$, which is the solution obtained for $f_i^n \equiv 0$, $u_0(x) \equiv u_1(x) \equiv 0$. This yields the constant coefficient scheme

$$\text{(3.12)} \quad \begin{aligned} &\text{(a)} \quad \delta^+ y_{1i}^n = \Delta y_{2i}^n + \frac{k}{2}\,[\phi'(0)\Delta^+\Delta^- y_{1i}^n - \psi'(0)\Delta^+\Delta^-(b \overset{k}{*} y_{1i}^n)] + g_{1i}^n \\ &\text{(b)} \quad \delta^+ y_{2i}^n = \phi'(0)\Delta y_{1i}^n - \psi'(0)\Delta(b \overset{k}{*} y_{1i}^n) \\ &\qquad + \frac{k}{2}\,[\phi'(0)\Delta^+\Delta^- y_{2i}^n - b(0)\psi'(0)\Delta y_{1i}^n - \psi'(0)\Delta(b' \overset{k}{*} y_{1i}^n)] + g_{2i}^n \end{aligned}$$

for $n > 0$, $i \in \mathbb{Z}$, with the initial data

$$\text{(3.13)} \quad \begin{aligned} &\text{(a)} \quad y_{1i}^0 = y_{10i}\,, \quad i \in \mathbb{Z} \\ &\text{(b)} \quad y_{2i}^0 = y_{20i}\,, \quad i \in \mathbb{Z}\,. \end{aligned}$$

The stability analysis will proceed in the discrete L^2-space

(3.14) $$L_h^2 = \{\tilde{f} = (f_j)_{j\in\mathbb{Z}} | f_j \in \mathbb{R}, \|\tilde{f}\|_h = (h \sum_{j\in\mathbb{Z}} |f_j|^2)^{1/2} < \infty\}$$

(Once stability in L^2 is known, it is easy to deduce stability is higher order discrete Sobolev spaces).

We assume that the initial data in (3.13) and the inhomogeneous terms in (3.12) take values in L_h^2.

Since we wish to apply Fourier analysis, we extend (3.12) to the whole real line by defining $L^2(\mathbb{R})$-functions g_1^n, g_2^n, y_{10}, y_{20} such that they take the prescribed values at the grid points. We then regard (3.12), (3.13) as equations on all of $\mathbb{R}$. The spatial difference operators are defined for $L^2(\mathbb{R})$-functions in the obvious way

$$(\Delta^+ y)(x) = \frac{y(x+h) - y(x)}{h} \text{ etc.}$$

For $y \in L^2(\mathbb{R})$ we denote the Fourier transform by $\hat{y}$:

(3.15) $$y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixs}\hat{y}(s)ds$$

Now we prove

Lemma 3.1.

Let $0 < \mu = \frac{k}{h}\sqrt{\phi'(0)} < 1$ and assume that $\phi'(0) > 0$, $b, b' \in L^1(\mathbb{R}^+)$, and (3.9) holds. Then the stability estimate

(3.16) $$\begin{aligned} &\max_{j=0(1)n} (\|\tilde{y}_1^j\|_{L_h^2} + \|\tilde{y}_2^j\|_{L_h^2}) < \\ &< C(t_n)(\|\tilde{y}_{10}\|_{L_h^2} + \|\tilde{y}_{20}\|_{L_h^2} + \max_{j=0(1)n-1} (\|\tilde{g}_1^j\|_{L_h^2} + \|\tilde{g}_2^j\|_{L_h^2})) \end{aligned}$$

holds for h sufficiently small. Here $\tilde{y}_1^j, \tilde{y}_2^j$ denotes the solution of (3.12), (3.13).

Proof. We apply Fourier transform and set $\hat{Y}^n = \begin{pmatrix} \hat{y}_1^n \\ \hat{y}_2^n \end{pmatrix}$, $\hat{G}^n = \begin{pmatrix} \hat{g}_1^n \\ \hat{g}_2^n \end{pmatrix}$, $\omega = sh$. This yields the equation

$$(3.17)\qquad \hat{Y}^{n+1}(s) = B(\omega)\hat{Y}^{n}(s) - \psi'(0)\begin{pmatrix} \dfrac{\mu^2}{\phi'(0)}(\cos\omega - 1) \\ i\dfrac{\mu}{\sqrt{\phi'(0)}}\sin\omega \end{pmatrix}(b \overset{k}{*} \hat{y}_1^n(s))$$

$$-ik\frac{\mu\psi'(0)}{\sqrt{\phi'(0)}}\binom{0}{\sin\omega}(b' \overset{k}{*} \hat{y}_1^n(s) + b(0)\hat{y}_1^n(s)) + k\hat{G}^n(s)$$

where

$$(3.18)\qquad B(\omega) = I + i\frac{\mu}{\sqrt{\phi'(0)}}A\sin\omega + \frac{\mu^2}{\phi'(0)}A^2(\cos\omega - 1),$$

$$A = \begin{pmatrix} 0 & 1 \\ \phi'(0) & 0 \end{pmatrix}.$$

We transform A to its diagonal form

$$(3.19)\qquad A = E\,\mathrm{diag}(\sqrt{\phi'(0)}, -\sqrt{\phi'(0)})E^{-1},\quad E = \begin{pmatrix} 1 & 1 \\ \sqrt{\phi'(0)} & -\sqrt{\phi'(0)} \end{pmatrix}$$

and set

$$(3.20)\qquad \hat{V}^n(s) = E^{-1}\hat{Y}^n(s)$$

We thus obtain the new difference scheme

$$(3.21)\qquad \hat{V}^{n+1}(s) = \mathrm{diag}(z(\omega),\bar{z}(\omega)) - \frac{\psi'(0)}{\phi'(0)}\binom{z(\omega) - 1}{\bar{z}(\omega) - 1}\Big(b \overset{k}{*} \Big(\frac{\hat{v}_1^n(s) + \hat{v}_2^n(s)}{2}\Big)\Big)$$

$$-ik\mu\frac{\psi'(0)}{\phi'(0)}\binom{\sin\omega}{-\sin\omega}\Big(b' \overset{k}{*} \Big(\frac{\hat{v}_1^n(s)+\hat{v}_2^n(s)}{2}\Big) + b(0)\,\frac{\hat{v}_1^n(s)+\hat{v}_2^n(s)}{2}\Big) + kE^{-1}\hat{G}^n(s)$$

where

$$(3.22)\qquad z(\omega) = 1 + i\mu\sin\omega + \mu^2(\cos\omega - 1)$$

An easy calculation shows that $|z(\omega)| < 1$, $\omega \neq 2\ell\pi$, $\ell \in \mathbb{Z}$, and $z(2\ell\pi) = 1$, provided that $0 < \mu < 1$. In the following, we only use that $|z(\omega)| < 1 + O(k)$. Hence, $|z(\omega)|^n$ is uniformly bounded for $n < \frac{c}{k}$. For stability on bounded intervals, we need not be concerned with $O(k)$-perturbations on the right side of (3.21) (see [19], section 3.9). It is thus sufficient to study the perturbed problem

$$(3.23)\qquad \hat{W}^{n+1} = \mathrm{diag}(z(\omega),\bar{z}(\omega))\hat{W}^n - \frac{\psi'(0)}{\phi'(0)}\binom{z(\omega) - 1}{\bar{z}(\omega) - 1}\Big(b \overset{k}{*} \Big(\frac{\hat{w}_1^n + \hat{w}_2^n}{2}\Big)\Big) + k\hat{H}^n$$

which we rewrite as

$$(3.24)\qquad \hat{w}^n = \mathrm{diag}(z^n(\omega),\bar{z}^n(\omega))\hat{w}^0 - \frac{\psi'(0)}{\phi'(0)} H\Big(\big(\frac{\bar{z}(\omega)-1}{z(\omega)-1}\big)\big(b \overset{k}{*} \big(\frac{\hat{w}_1^m + \hat{w}_2^m}{2}\big)\big)\Big)_{m=0}^{n-1} + k\,(\hat{H}^j)_{j=0}^{n-1} .$$

where the operator H is defined as

$$(3.25)\qquad H\big(\hat{G}^j\big)_{j=0}^{n} = \sum_{j=0}^{n} \mathrm{diag}(z^{n-j}(\omega),\bar{z}^{n-j}(\omega))\hat{G}^j$$

Summation by parts yields $\Big(\text{with } \hat{u}^j = \big(\frac{\hat{w}_1^j + \hat{w}_2^j}{2}\big)\Big)$

$$(3.26)\qquad \sum_{m=0}^{n-1} z^{n-1-m}(1-z)(b \overset{k}{*} \hat{u}^m) = (1-z^n)(b \overset{k}{*} \hat{u}^{n-1})$$

$$+ \sum_{m=0}^{n-2} (z^{n-m-1} - z^n)(b \overset{k}{*} \hat{u}^{m+1} - b \overset{k}{*} \hat{u}^m)$$

$$= b \overset{k}{*} \hat{u}^{n-1} - z^{n-1}\sum_{m=0}^{n-2} z^{-m}[k\alpha_{m+1,m+1} b_0 \hat{u}^{m+1}$$

$$+ k\sum_{j=0}^{m} (\alpha_{j,m+1} b_{m+1-j} - \alpha_{jm} b_{m-j})\hat{u}^j] - kz^n \alpha_{00} b_0 \hat{u}_0$$

Moreover, note that $\alpha_{j,m+1} b_{m+1-j} - \alpha_{jm} b_{m-j} = (\alpha_{j,m+1} - \alpha_{jm})\, b_{m+1-j} + \alpha_{jm}(b_{m+1-j} - b_{m-j})$ and that $b_{m+1-j} - b_{m-j} = O(k)$. Using (3.9), one easily obtains from (3.26)

$$(3.27)\qquad \Big\| H\Big(\big(\frac{\bar{z}(\omega)-1}{z(\omega)-1}\big)\big(b * \big(\frac{\hat{w}_1^m + \hat{w}_2^m}{2}\big)\big)\Big)_{m=0}^{n-1} \Big\| \le C\cdot t_n \max_{j=0(1)n-1} \|\hat{w}^n\| .$$

with some constant C. If we choose T such that $\gamma = CT < 1$, we get from (3.23) that

$$(3.28)\qquad \max_{j=0(1)n} \|\hat{w}^j\| \le \gamma \max_{j=0(1)n-1} \|\hat{w}^j\| + C_1\|\hat{w}^0\| + C_2 T \max_{j=0(1)n-1} \|\hat{H}^j\|$$

and hence

$$\max_{j=0(1)n} \|\hat{w}^j\| \le \frac{1}{1-\gamma}\,(c_1\|\hat{w}^0\| + c_2 T \max_{j=0(1)n-1} \|\hat{H}^j\|) \tag{3.29}$$

holds for $t_n \le T$.

This argument can be iterated. We may pose a new initial value problem at the grid point nearest to $t = T$ and continue the solution into the interval $[T,2T]$ etc. This yields an estimate of the form

$$\max_{kT\le t_j\le(k+1)T} \|\hat{w}^j\| \le \frac{1}{(1-\gamma)^{k+1}}\,(c_1\|\hat{w}_0\| + c_2 T \max_{j=0(1)n} \|\hat{H}^j\|) \tag{3.30}$$

for $k \in \mathbb{N}$. (3.16) immediately follows.

We see from this proof that solutions of the linearized problem can grow at most exponentially. The integral has been treated as a "lower order" perturbation and we made no use of the fact that it has a damping influence, in fact, no conditions on the sign of the kernel were needed here. A convergence statement reflecting the damping requires a more sophisticated analysis. For a special case this is dealt with in section 4.

As mentioned above, we have to study the linearization at the exact solution $v(x,t), w(x,t)$ of (3.2), (3.3) rather than the linearization at the trivial solution. This yields the scheme

(3.31)

$$\text{(a)}\quad y_{1i}^{n+1} = y_{1i}^n + k\Delta y_{2i}^n + \frac{k^2}{2}\,[\Delta^+\Delta^-(\phi'(v(x_i,t_n))y_{1i}^n) - \Delta^+\Delta^-(b \overset{k}{*} \phi'(v)y_{1i}^n)] + kg_{1i}^n$$

$$\begin{aligned}\text{(b)}\quad y_{2i}^{n+1} = y_{2i}^n &+ k[\Delta(\phi'(v)y_{1i}^n) - \Delta(b \overset{k}{*} (\phi'(v)y_{1i}^n)) \\ &+ \frac{k^2}{2}\,[\Delta^+(\phi'(\tfrac{1}{2}\,(v(x_i,t_n) + v(x_{i-1},t_n)))\Delta^- y_{2i}^n \\ &+ \frac{1}{2}\,\phi''(\tfrac{1}{2}\,(v(x_i,t_n) + v(x_{i-1},t_n)))(y_{1i}^n + y_{1,i-1}^n)\Delta^- w(x_i,t_n)) \\ &- b(0)\Delta(\phi'(v)y_{1i}^n) - b' \overset{k}{*} (\phi'(v)y_{1i}^n)] + kg_{2i}^n\end{aligned}$$

(3.32)

$$\text{(a)}\quad y_{1i}^0 = y_{10i}$$

$$\text{(b)}\quad y_{2i}^0 = y_{20i}$$

Lemma 3.2.

Assume $0 < C < \frac{k}{h} \max_{x \in \mathbb{R}, t \in [0,\tau]} \sqrt{\phi'(v(x,t))} < 1$ holds and assume that b, ϕ, ψ are smooth, $b, b' \in L^1(\mathbb{R}^+)$, $\phi' > 0$. If the exact solution v, w is smooth on $\mathbb{R} \times [0,\tau]$, then an estimate of the form (3.16) holds for the solution of (3.32), (3.32) on $[0,\tau]$.

Proof.

Again we may neglect terms of order k on the right hand side of (3.31), which leaves us with the perturbed problem

$$(3.33)\quad \begin{aligned} &(a)\quad v_{1i}^{n+1} = v_{1i}^n + k\Delta v_{2i}^n + \frac{k^2}{2}\phi_i^n \Delta^+\Delta^- v_{1i}^n - \frac{k^2}{2}\Delta^+\Delta^- b \overset{k}{*} (\psi_i^n v_{1i}^n) + kh_{1i}^n \\ &(b)\quad v_{2i}^{n+1} = v_{2i}^n + k\phi_i^n \Delta v_{1i}^n + \frac{k^2}{2}\phi_i^n \Delta^+\Delta^- v_{2i}^n - k\Delta b \overset{k}{*} (\psi_i^n v_{1i}^n) + kh_{2i}^n \end{aligned}$$

where $\phi_i^n = \phi'(v(x_i,t_n))$, $\psi_i^n = \psi'(v(x_i,t_n))$. With

$$(3.34)\quad A_i^n = \begin{pmatrix} 0 & 1 \\ \phi_i^n & 0 \end{pmatrix}, \quad V_i^n = \begin{pmatrix} v_{1i}^n \\ v_{2i}^n \end{pmatrix}, \quad H_i^n = \begin{pmatrix} h_{1i}^n \\ h_{2i}^n \end{pmatrix},$$

(3.33) reads

$$(3.35)\quad V_i^{n+1} = V_i^n + kA_i^n \Delta V_i^n + \frac{k^2}{2}(A_i^n)^2 \Delta^+\Delta^- V_i^n - \begin{pmatrix} \frac{k^2}{2}\Delta^+\Delta^-(b \overset{k}{*} (\psi_i^n v_{1i}^n)) \\ k\Delta(b \overset{k}{*} (\psi_i^n v_{1i}^n)) \end{pmatrix} + kH_i^n$$

As before, we diagonalize A_i^n by setting

$$(3.36)\quad E_i^n = \begin{pmatrix} 1 & 1 \\ \sqrt{\phi_i^n} & -\sqrt{\phi_i^n} \end{pmatrix}, \quad J_i^n = \begin{pmatrix} \sqrt{\phi_i^n} & 0 \\ 0 & -\sqrt{\phi_i^n} \end{pmatrix}$$

so that $A_i^n = E_i^n J_i^n (E_i^n)^{-1}$. We substitute

$$(3.37)\quad V_i^n = E_i^n U_i^n$$

thus obtaining (after extending all grid functions to real functions:

$$(3.38) \qquad U^{n+1}(x) = (L_n U^n)(x) + (I - L_n)\cdot\binom{1}{1}\cdot \frac{1}{\phi^n(x)} \cdot\left(b \overset{k}{*} (\psi^n(x))\cdot\left(\frac{u_1^n(x)+u_2^n(x)}{2}\right)\right)$$

$$+ kH^n(x) + O(k)$$

where $L_n : (L^2(\mathbb{R}))^2 \to (L^2(\mathbb{R}))^2$ is defined as

$$(3.39) \qquad L_n U(x) = U(x) + kJ^n(x)\Delta U(x) + \frac{k^2}{2}(J^n(x))^2 \Delta^+\Delta^- U(x) \; .$$

Equation (3.38) can be rewritten as

$$U^n = \sum_{m=0}^{n-1} (L_{n-1}L_{n-2} \cdots L_{m+1})(I - L_m)\cdot\binom{1}{1}\cdot \frac{1}{\phi^m(x)} \cdot$$

$$\cdot \left(b \overset{k}{*} (\psi^m(x))\left(\frac{U_1^m(x) + U_2^m(x)}{2}\right)\right) + (L_{n-1} \cdots L_1)U^0$$

$$+ k \sum_{m=0}^{n-1} (L_{n-1}L_{n-2} \cdots L_{m+1})H^m$$

Partial summation yields

$$(3.41) \qquad \sum_{m=0}^{n-1} (L_{n-1} \cdots L_{m+1})(I - L_m)p_m = (I - L_{n-1}L_{n-2} \cdots L_0)p_{n-1}$$

$$= \sum_{m=0}^{n-2} L_{n-1} \cdots L_{m+1}(I - L_m \cdots L_0)(p_{m+1} - p_m)$$

The "local" amplification matrix of L_n is given by

$$(3.42) \qquad Z(x,\omega) = I + iJ^n(x)\frac{k}{h}\sin\omega + (J^n(x))^2 \frac{k^2}{h^2}(\cos\omega - 1)$$

where ω has the same meaning as before. If $\frac{k}{h}\sqrt{\phi^n(x)} \le \lambda < 1$ for $x \in \mathbb{R}$, then $\|Z(x,\omega)\| \le 1$ for $x,\omega \in \mathbb{R}$. It follows from the theorem on page 121 in [19] that

$$\|L_n\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le 1 + Ck \tag{3.43}$$

where $C > 0$ can be chosen independently of n,k for $t_n \in [0,\tau]$. The rest of the proof proceeds in the same manner as for lemma 3.1.

Using Keller's [11] nonlinear stability-consistency principle, we obtain the following convergence result:

<u>Theorem 3.1.</u>

Let b,ϕ,ψ be sufficiently smooth, $b,b' \in L^1(\mathbb{R}^+)$, $\phi' > 0$. Also assume that the data f,u_0,u_1 are smooth. (As a consequence, (3.2), (3.3) has a smooth solution v,w). Choose the scheme (3.6), (3.10) such that $0 < c < \frac{k}{h} \max_{x\in\mathbb{R},t\in[0,\tau]} \sqrt{\phi'(v(x,t))} < 1$ holds and such that (3.9), (3.11) are satisfied. Then there is a unique solution $\tilde{v}^n = (v_i^n)_{i\in\mathbb{Z}}$, $\tilde{w}^n = (w_i^n)_{i\in\mathbb{Z}} \in L_h^2$ of (3.6), (3.10), which fulfills the convergence estimate

$$\|(v_i^n - v(x_i,t_n))_{i\in\mathbb{Z}}\|_{L_h^2} + \|(w_i^n - w(x_i,t_n))_{i\in\mathbb{Z}}\|_{L_h^2} \le Dh^2 \tag{3.44}$$

where D can be chosen independent of h, $t_n \le \tau$.

By applying the same analysis to the "differenced" equations, convergence estimates in higher Sobolev norms can be obtained.

It is clear that the numerical performance of the method very much depends on how "well" the second order accurate integration rule (3.8) integrates b and b' and on how large derivatives of v and w get. Particular care must be taken for a class of practically important kernels with the following behavior

$$|b^{(i)}(\sigma)| \approx \frac{A}{\varepsilon^i} e^{-\sigma/\varepsilon}, \quad i \in \mathbb{N}, \quad 0 < \varepsilon \ll 1, \quad A > 0 . \tag{3.45}$$

The trapezoidal rule approximation

$$\int_0^t |b'(\sigma)|d\sigma \approx \frac{kA}{\varepsilon}\left(\frac{1}{2} + \frac{1}{1 - e^{-k/\varepsilon}} + O(e^{-t/\varepsilon})\right) \tag{3.46}$$

in unstable unless $\frac{k}{\epsilon} <$ const. and the explicit scheme (3.6), (3.10) also requires this serious restriction.

In the next chapter, we shall deal with the special case where the kernel is a sum of exponentials. We show in this case how stiffness problems can be avoided. Also, we compute the convolutions recursively, which is less time consuming than using formula (3.8).

4. GLOBAL ANALYSIS

We assume that the kernel is of the form (1.4) and set

$$v = u_x, w = u_t, z_\ell = K_\ell \int_0^t e^{-\lambda_\ell(t-\tau)} \psi(u_x(x,\tau))_x \, d\tau \tag{4.1}$$

Then (1.1) is equivalent to the system

$$\begin{aligned} v_t &= w_x \\ w_t &= \phi(v)_x - \sum_{\ell=1}^{M} z_\ell + f(x,t) \\ z_{\ell t} &= K_\ell \psi(v)_x - \lambda_\ell z_\ell \end{aligned} \tag{4.2}$$

and the initial conditions $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$ become

$$v(x,0) = u_0'(x), \quad w(x,0) = u_1(x), \quad z_\ell(x,0) = 0 \tag{4.3}$$

Following the recipe (3.5), we obtain the following Lax-Wendroff discretization for (4.2)

$$\begin{aligned} v_i^{n+1} &= v_i^n + k\Delta w_i^n + \frac{k^2}{2}[\Delta^+\Delta^-\phi(v_i^n) - \Delta \sum_{\ell=1}^{M} z_{\ell i}^n + \Delta f_i^n] \\ w_i^{n+1} &= w_i^n + k[\Delta\phi(v_i^n) - \sum_{\ell=1}^{M} z_{\ell i}^n + f_i^n] \\ &\quad + \frac{k^2}{2}[\Delta^+(\phi'(\tfrac{1}{2}(v_i^n + v_{i-1}^n))\Delta^- w_i^n) - \sum_{\ell=1}^{M} K_\ell \Delta\psi(v_i^n) + \sum_{\ell=1}^{M} \lambda_\ell z_{\ell i}^n + f_{ti}^n] \\ z_{\ell i}^{n+1} &= z_{\ell i}^n + k[K_\ell\Delta\psi(v_i^n) - \lambda_\ell z_{\ell i}^n] + \\ &\quad + \frac{k^2}{2}[K_\ell\Delta^+(\psi'(\tfrac{1}{2}(v_i^n + v_{i-1}^n))\Delta^- w_i^n) - \lambda_\ell(K_\ell\Delta\psi(v_i^n) - \lambda_\ell z_{\ell i}^n)] \end{aligned} \tag{4.4}$$

for $i \in \mathbb{Z}$, $n > 0$, subject to the initial conditions

$$v_i^0 = u_0'(x_i), \; w_i^0 = u_1(x_i), \; z_{\ell i}^0 = 0 \tag{4.5}$$

Here v_i^n denotes the approximation to $v(x_i,t_n)$ etc. The last equation of (4.4) is stiff for $\lambda_\ell \gg 1$. We obtain the growth function

$$\omega_\ell(\lambda_\ell k) = 1 - \lambda_\ell k + \frac{1}{2}(\lambda_\ell k)^2 , \tag{4.6}$$

which fulfills $|\omega_\ell| < 1$ iff $0 < \lambda_\ell k < 2$ holds. The scheme (4.4) will thus be unstable unless $k \max_\ell \lambda_\ell < 2$ holds (for some rheological applications λ_ℓ may be as large as 10^6 [22]!). However, this can be repaired by making the scheme (4.4) semi-implicit, e.g. by replacing the term $\frac{k^2}{2} \lambda_\ell^2 z_{\ell i}^n$ by $\frac{k^2}{2} \lambda_\ell^2 z_{\ell i}^{n+1}$. In that case we get

$$(4.7) \qquad \omega_\ell(\lambda_\ell k) = \frac{1 - \lambda_\ell k}{1 - \frac{1}{2}(\lambda_\ell k)^2}$$

and $|\omega_\ell| < 1$ if $0 < \lambda_\ell k < -1 + \sqrt{5}$ or $\lambda_\ell k > 2$. Moreover, we have $0 < \omega_\ell < 1$ for $0 < \lambda_\ell k < 1$ or $\lambda_\ell k > 2$. The growth function of the fully implicit scheme is negative (but less than one in modulus) for $\lambda_\ell k > 1 + \sqrt{5}$. Since this produces oscillations in the numerical solution, the semi-implicit scheme is to be preferred.

The convergence of the scheme (4.4) is analyzed in the same fashion as in section 3, the nontrivial step again being the stability proof. In the present situation, stability must be shown globally in time, and not only on finite time intervals. We assume the situation given in section 2, i.e. $\phi = \psi$, solutions are spatially 2π-periodic (of course the mesh size is chosen as an integral divisor of the period, and f, u_0, u_1 are small. Rather than studying the stability of (4.4), we investigate the Lax-Wendroff discretization of the symmetric hyperbolic form (2.2). Although the two schemes are not equivalent, they only differ by higher order terms negligible for the stability analysis. Equation (2.2) leads to the following scheme:

$$
\begin{aligned}
v_i^{n+1} &= v_i^n + k\beta(v_i^n)\Delta w_i^n + \frac{k^2}{2}\Big[\beta(v_i^n)\{\Delta^+(\beta(\tfrac{1}{2}v_i^n + \tfrac{1}{2}v_{i-1}^n)\Delta^- v_i^n) \\
&\quad - \Delta\Big(\sum_{m=1}^{M} K_m(g_{mi}^n + w_i^n) + f_i^n\Big)\} + \beta'(v_i^n)\beta(v_i^n)(\Delta w_i^n)^2\Big] \\
w_i^{n+1} &= w_i^n + k\Big[\beta(v_i^n)\Delta v_i^n - \sum_{m=1}^{M} K_m(g_{mi}^n + w_i^n) + f_i^n\Big] \\
&\quad + \frac{k^2}{2}\Big[\beta(v_i^n)\Delta^+(\beta(\tfrac{1}{2}v_i^n + \tfrac{1}{2}v_{i-1}^n)\Delta^- w_i^n) + \beta'(v_i^n)\beta(v_i^n)\Delta v_i^n\cdot\Delta w_i^n \\
&\quad - \sum_{m=1}^{M} K_m\{-\lambda_m(g_{mi}^n + w_i^n) + \beta(v_i^n)\Delta v_i^n\} + f_{ti}^n\Big] \\
g_{\ell i}^{n+1} &= g_{\ell i}^n + k\Big[-\lambda_\ell(g_{\ell i}^n + w_i^n) + \sum_{m=1}^{M} K_m(g_{mi}^n + w_i^n) - f_i^n\Big] \\
&\quad + \frac{k^2}{2}\Big[+\lambda_\ell^2(g_{\ell i}^n + w_i^n) - \lambda_\ell\beta(v_i^n)\Delta v_i^n + \sum_{m=1}^{M} K_m\{-\lambda_m(g_{mi}^n + w_i^n) + \beta(v_i^n)\Delta v_i^n\} - f_{ti}^n\Big]
\end{aligned}
\tag{4.7}
$$

We first deal with the stability of the trivial solution $v = w = g_\ell = 0$. The linearization of the right hand side at this solution has the form

$$L = I + kL_0 + kL_1\Delta + \frac{k^2}{2}(L_1\Delta^+L_1\Delta^- + L_0L_1\Delta + L_1L_0\Delta + L_0^2)$$

where L_0, L_1 are the constant coefficient matrices

$$
L_1 = \begin{pmatrix}
0 & \beta(0) & 0 \; . \; . \; . \; 0 \\
\beta(0) & 0 & 0 \; . \; . \; . \; 0 \\
0 & 0 & \\
\vdots & \vdots & 0 \\
0 & 0 &
\end{pmatrix}
$$

$$
L_0 = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & -\sum_{m=1}^{M} K_M & -K_1 & \cdots & -K_M \\
0 & -\lambda_1 + \sum_{m=1}^{M} K_m & -\lambda_1 + K_1 & \cdots & K_M \\
\vdots & \vdots & \vdots & & \vdots \\
0 & -\lambda_M + \sum_{m=1}^{M} K_m & +K_1 & \cdots & -\lambda_M + K_M
\end{pmatrix}
$$

We expand v, w, g_ℓ in Fourier series:

$$v_i^n = \sum_{K=-[\frac{\pi}{h}]}^{[\pi/h]} v_i^n(K) e^{iKh}$$

Then, for the Kth Fourier component, the operator Δ becomes $i \frac{\sin(Kh)}{h}$ and $\Delta^+\Delta^-$ becomes $2 \frac{\cos(Kh) - 1}{h^2}$. We thus have to investigate the spectrum of the matrix

$$L(K) = I + kL_0 + \frac{k}{h} i \sin(Kh) L_1 + \frac{k^2}{h^2} (\cos(Kh) - 1)$$

$$\cdot L_1^2 + \frac{k^2}{2h} \sin(Kh) \; (L_0L_1 + L_1L_0) + \frac{k^2}{2} L_0$$

For the following, we assume

(4.8) $$\frac{k}{h} \beta(0) = \text{const.} < 1 .$$

If $K \neq 0$ is fixed, then $L(K) = I + k(L_0 + KL_1) + O(k^2)$. Since the eigenvalues of $L_0 + KL_1$ have negative real parts (cf. section 2), the eigenvalues of $L(K)$ are inside the unit circle. For $K = 0$, $L(K)$ has a double eigenvalue one, the remaining eigenvalues are inside the unit circle. The eigenvalue one may be transformed away in a fashion analogous to section 2 and need not concern us further.

We have to investigate the behaviour of $L(K)$ as $K \to \infty$. In this case, at least one of the terms $\sin(Kh)$, $\cos(Kh) - 1$ is large compared to h. The eigenvalues of $L(K)$ are in first order given by those of

$$I + \frac{k}{h} i \sin(Kh) L_1 + \frac{k^2}{h^2} (\cos(Kh) - 1) L_1^2 .$$

This operator has an M-fold eigenvalue 1 and the simple eigenvalues $1 \pm i \frac{k}{h} \beta(0) \sin(Kh) + \frac{k^2}{h^2} \beta^2(0)(\cos(Kh) - 1)$. If (4.8) holds, these two eigenvalues lie on an ellipse inside the unit circle. A simple perturbation analysis shows that the M-fold eigenvalue one is perturbed into M distinct eigenvalues inside the unit circle, and their distance from the unit circle is of order k. As in section 2, there is a matrix

$T(K)$ such that $T^{-1}(K)L(K)T(K)$ is dissipative; $T(K), T^{-1}(K)$ are bounded independently of K, and for K large $T(K)$ has the form (2.5). Thus the trivial solution is stable.

As in section 2, we must show stability for the variable coefficient problem when v is in a neighborhood of 0. There, we made use of the fact that the principal part of the differential operator was skew-adjoint. In the discrete case, we have to investigate the operator

$$L(v_i) = I + kL_1(v_i)\Delta + \frac{k^2}{2} L_1(v_i)\Delta^+ L_1(\frac{1}{2} v_i + \frac{1}{2} v_{i-1})\Delta^-$$

$$\text{where } L_1(v) = \begin{pmatrix} 0 & \beta(v) & 0 \; . \; . \; . \; 0 \\ \beta(v) & 0 & 0 \; . \; . \; . \; 0 \\ 0 & 0 & \\ \vdots & & \\ 0 & 0 & \end{pmatrix} .$$

All other contributions to the linearization are of order k. Let us put $A = L_1(v_i)\Delta^+$. Then the adjoint of A is $A^* = -\Delta^- L_1(v_i)$ and it is immediately verified that

$$L_1(v_i) = I + \frac{k}{2}(A - A^*) - \frac{k^2}{4}(AA^* + A^*A) + O(k\|v\|) .$$

A simple calculation yields

$$\|(I + \frac{k}{2}(A - A^*) - \frac{k^2}{4}(AA^* + A^*A))z\|^2 = \|z\|^2 - \frac{1}{4} k^2 \|(A + A^*)z\|^2 + O(\frac{k}{h})^3 .$$

If $\frac{k}{h}$ is chosen sufficiently small (but still of order 1), then an argument similar to section 2 guarantees the stability of the variable coefficient problem. We thus arrive at the following result.

Theorem 4.1.

Assume that the assumption of section 2 holds and that $\frac{k}{h}$ is chosen small enough. Then the convergence estimate (3.44) holds uniformly in time.

5. NUMERICAL EXPERIMENTS

For all computations we used a kernel b of the form (1.4) and we employed the scheme given by (4.4) and (4.5) with the semi-implicit modification. The spatial mesh size was prescribed and the temporal mesh size was determined at each step such that the stability conditions were satisfied. In particular, we chose $\frac{k}{h} \max_i \sqrt{\phi'(v_i^n)} < 0.8$. The calculations were performed at the VAX of the Mathematics Research Center, University of Wisconsin-Madison in double precision arithmetic.

To solve the initial value problem numerically, we introduced artificial (far out) boundaries at $x = \pm X$, and the boundary conditions $v(\pm X,t) = w(\pm X,t) = 0$ (since the initial data vanish at $\pm\infty$). This introduces an additional error of order $O(\max_{t\in[0,T]} |v(\pm X,t)| + \max_{t\in[0,T]} |w(\pm X,t)|)$. In all the computations reported below, this quantity can safely be neglected. We performed a convergence test for the problem (3.2), (3.3) with $\phi(v) = \psi(v) = v + \frac{1}{3} v^3, b(\sigma) = 0.4e^{-\sigma}$, where $v(x,0)$, $w(x,0)$, $f(x,t)$ were chosen such that the problem has the exact solution $v(x,t) = (1 - x^2)e^{\frac{x^2}{2}-t}$, $w(x,t) = -xe^{\frac{x^2}{2}-t}$. Table 1 shows the errors e_v, e_w (in the discrete L^2-norm) of v and w, resp. and the corresponding convergence rate given by $\ln\left(\frac{e(h)}{e(h/2)}\right)/\ln 2$ at two different t-values and for the maximal errors for $t \in [0,1]$. Obviously, the scheme is second order accurate for this smooth solution.

Table 2 shows that the L^2-errors of v and w decay as t increases. The reason for this is the dissipative effect of the Volterra term.

The following calculations were done using

$$\phi(v) = \psi(v) = 2v + 5v^2 + 25v^3 \tag{5.1}$$

$$b(\sigma) = 0.4e^{-\sigma} + 0.2e^{-2\sigma} \tag{5.2}$$

and the initial data

(5.3) (a) $$v_\varepsilon(x,0) = \varepsilon v_0(x) = \varepsilon(1 - 3x - x^2 + x^3)e^{-x^2/2}$$

(b) $$w_\varepsilon(x,0) = \varepsilon w_0(x) = \varepsilon(1 - x^2)e^{-x^2/2}$$

with $0 < \varepsilon \leq 1$. The force $f(x,t)$ was set equal to zero. ϕ' is strictly positive for all $v \in \mathbb{R}$. Figures 1 and 2 show $v_0(x)$, $w_0(x)$ resp., and the next plots show t-sections of v and w, i.e. they show v and w as functions of x for fixed t-values. For $\varepsilon = 1$ ("large" initial data) the dissipative influence of the Volterra term is not strong enough to avoid singularities. Figures 3-5 and 6-8 show the evolution of v and w resp. A shock-type singularity appears at $t = 0.057$, $x = 0.4$. Then a second ($t = 0.073$, $x = -1.6$) and a third ($t = 0.12$, $x = 2$) shock develops.

At this point we want to remind the reader that the existence of shocks for equation (1.1) has not been proved yet, actually there is no theory of weak solutions at all. However, it has been shown that smooth solutions may cease to exist after a finite time [8], because v_x and w_t tend to infinity.

Table 3 shows the maximal values of the difference quotients Δv_i^n, Δw_i^n for $t_n = 1$ and $i < 0$ (corresponding to the singularity at $x = -1.7$). Halving the mesh size approximately doubles these values, which means that the differences $v_{i+1}^n - v_i^n$, $w_{i+1}^n - w_i^n$ within the shock are practically independent of the mesh size.

Since the Lax-Wendroff method is an artificial viscosity method, we cannot expect completely sharp shocks. A shock layer of thickness $O(h)$ develops around the shock ([19], section 12.14). This is illustrated by Figures 10-14, which show the left shock in Fig. 9 for various mesh sizes. We have $\varepsilon = \frac{1}{5}$ and $t = 0.631$. The width of the shock layer is (constantly) about 3h (grid points are marked).

The "overshoot" (see also Figures 3-8) is typical for Lax-Wendroff method ([19], section 12.14) and is due to artificial dispersion. The high wavenumber components of the solution have a smaller wave speed and thus lag behind the shock front. The width of the "overshoot layer" decreases with h. Outside the shock-layer and overshoot region the solutions coincide up to the plot accuracy for $h = 0.01$, $h = 0.02$, $h = 0.04$.

Our convergence discussion in the previous section does of course not apply to solutions with singularities, but it is clear that, if the Lax-Wendroff method converges boundedly almost everywhere, then it converges to a weak solution [19].

Therefore the presented numerical evidence indicates that weak solutions u of (1.1), such that u_t and u_x have shocks, exist.

For decreasing ε the relative effect of dissipation becomes stronger. For $\varepsilon = \frac{1}{20}$ the *breakdown of smooth solutions occurs* at $t = 2.7$, while the second derivatives of the solution of the corresponding quasilinear wave equation

$$u_{tt} = \phi(u_x)_x \tag{5.4}$$

(with the same initial data) blow up already at $t = 2.1$.

Figures 16-21 show the evolution of v and w for $\varepsilon = \frac{1}{40}$. No singularities occur for $t \in [0,20]$, the dissipative mechanism of the Volterra term seems to produce globally smooth solutions here. In the Figures 15-21, L2N denotes the L^2-norm of v,w at the given time t. The decay of the L^2-norms with increasing t is shown in Table 4. It is clear from Figures 15-21 that the L^∞-norms also tend to zero as $t \to \infty$.

Figures 21-26 show the corresponding plot for the wave equation (5.4) (without the integral term). The first derivatives of v and w blow up at $t = 4.7$, and the energy given by (1.3) of course remains constant for $t > 0$ (apart from artificial viscosity effects which show up in the fourth digit of the L^2-norm).

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PM/MR/ed

		$e_v(h)$	Rate	$e_w(h)$	Rate
	$h = 0.1$	4.7439746×10^{-3}		2.5703589×10^{-3}	
$t = 0.49$	$h = 0.05$	1.2233024×10^{-3}	1.96	5.916561×10^{-4}	2.12
	$h = 0.025$	3.091415×10^{-4}	1.98	1.432932×10^{-4}	2.05
	$h = 0.1$	6.1822914×10^{-3}		2.2217676×10^{-3}	
$t = 0.96$	$h = 0.05$	1.5563728×10^{-3}	2.00	5.859864×10^{-4}	1.92
	$h = 0.025$	3.883662×10^{-4}	2.00	1.553665×10^{-4}	1.92
	$h = 0.1$	6.198451×10^{-3}		3.3348472×10^{-3}	
$\max_{t\in[0,1]}$	$h = 0.05$	1.5611899×10^{-3}	1.99	7.615021×10^{-4}	2.13
	$h = 0.025$	3.906411×10^{-4}	2.00	1.809902×10^{-4}	2.07

Table 1. Errors and Convergence Rates

t	$e_v(h = 0.1)$	$e_w(h = 0.2)$
1.	6.0477392×10^{-3}	2.6280316×10^{-3}
3.	3.3230997×10^{-3}	2.6443695×10^{-3}
5.	2.1187529×10^{-3}	1.9864532×10^{-3}
7.	1.4750333×10^{-3}	1.4025019×10^{-3}
9.	1.0429657×10^{-3}	9.730021×10^{-4}
11.	6.102893×10^{-4}	5.120333×10^{-4}
13.	2.335227×10^{-4}	1.883220×10^{-4}
15.	2.232786×10^{-4}	1.348227×10^{-4}

Table 2. Decay of Errors

	Δv	Δw
$h = 0.1$	8.748	53.19
$h = 0.05$	17.90	113.8
$h = 0.025$	34.41	223.6

Table 3. Numerically obtained values for v_x, w_x at $t = 1$, $x = -1.7$

t	L2N : V	L2N : W
0.	5.3925×10^{-2}	2.88243×10^{-2}
5.	1.20131×10^{-2}	1.60351×10^{-2}
10.	3.7581×10^{-3}	4.81279×10^{-3}
15.	1.5276×10^{-3}	1.8037×10^{-3}

Table 4. Decay of L^2-norms due to dissipation

Figure 1. V_0

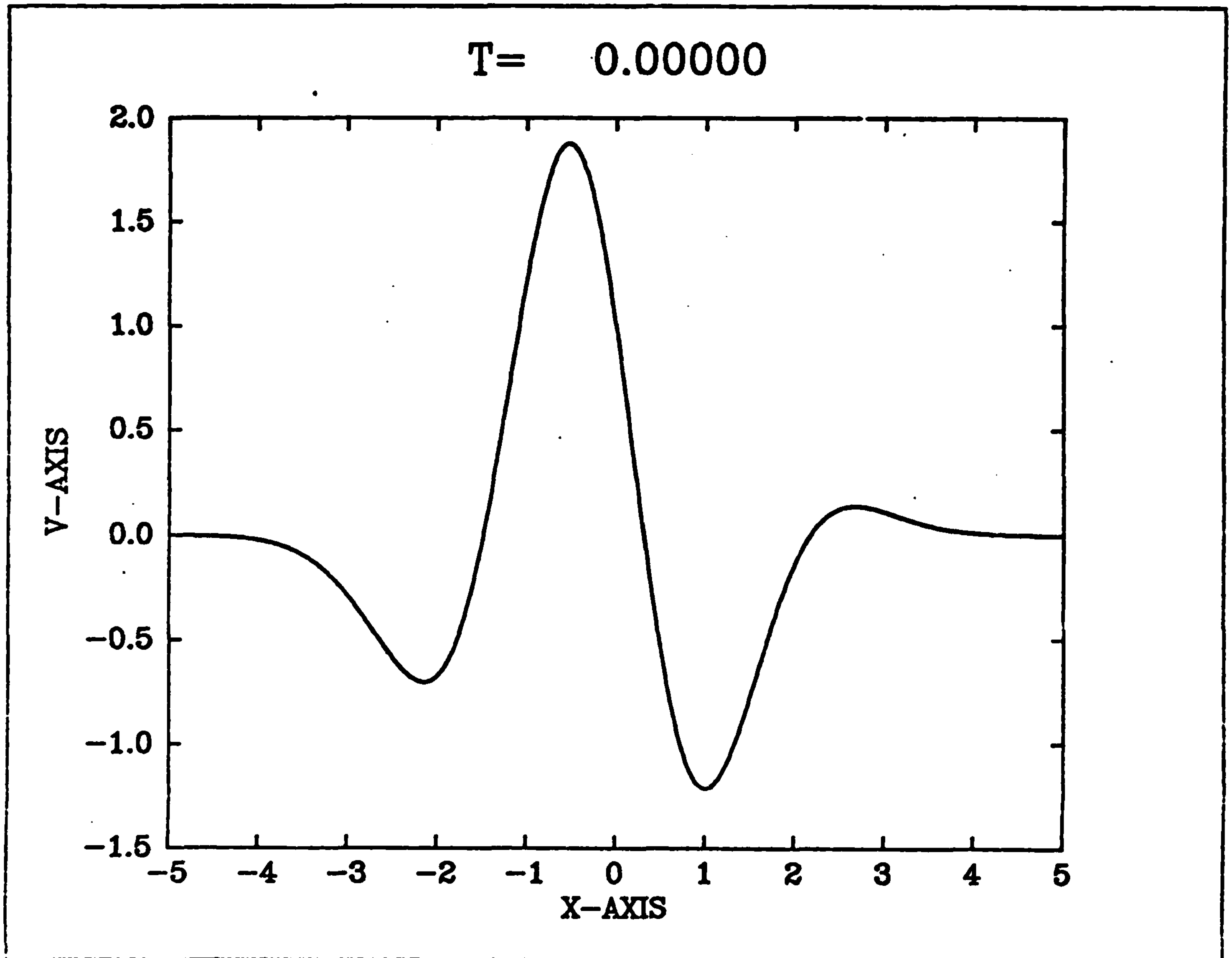


Figure 2. W_0

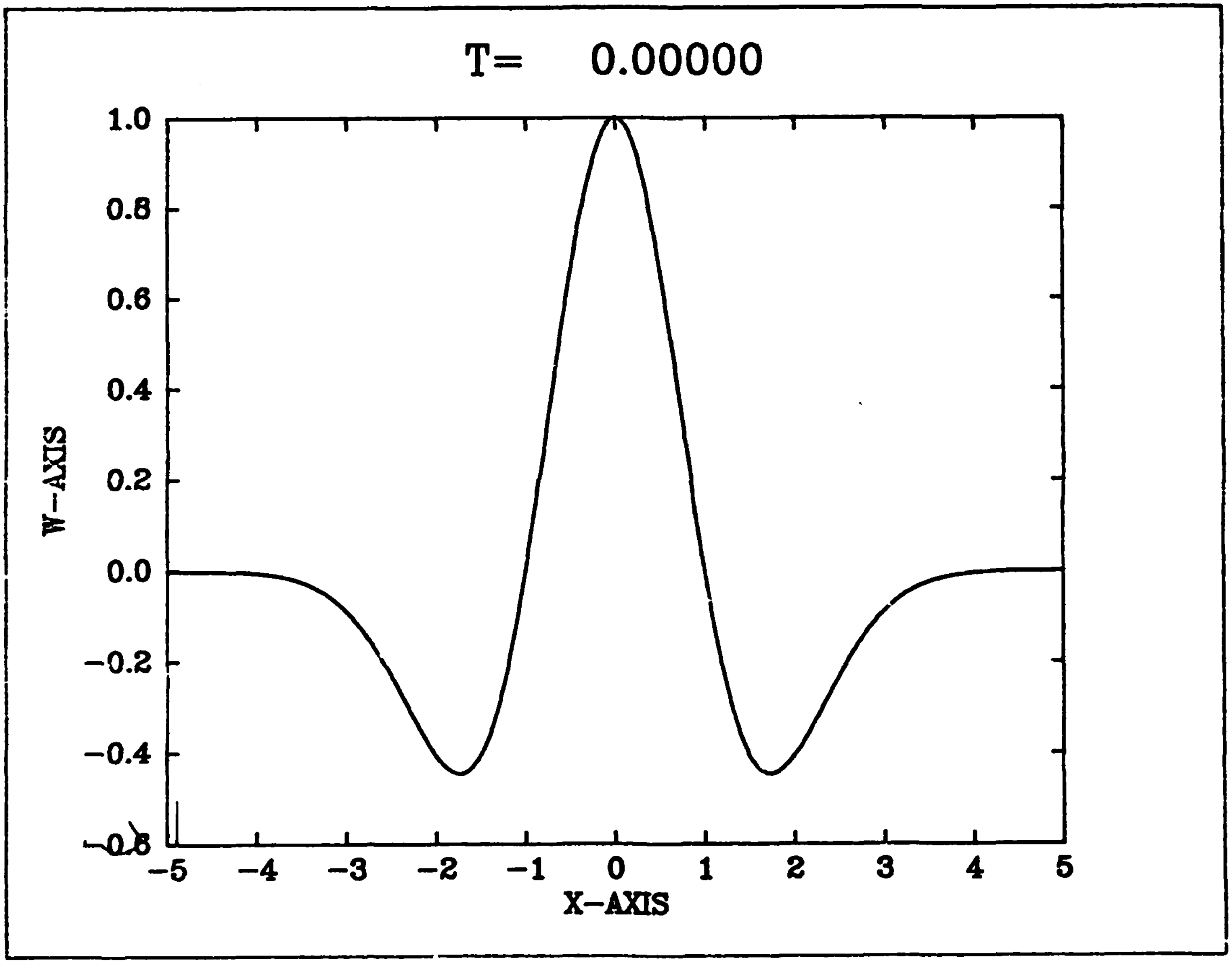


Figure 3. Development of shocks for large data, $\varepsilon = 1$

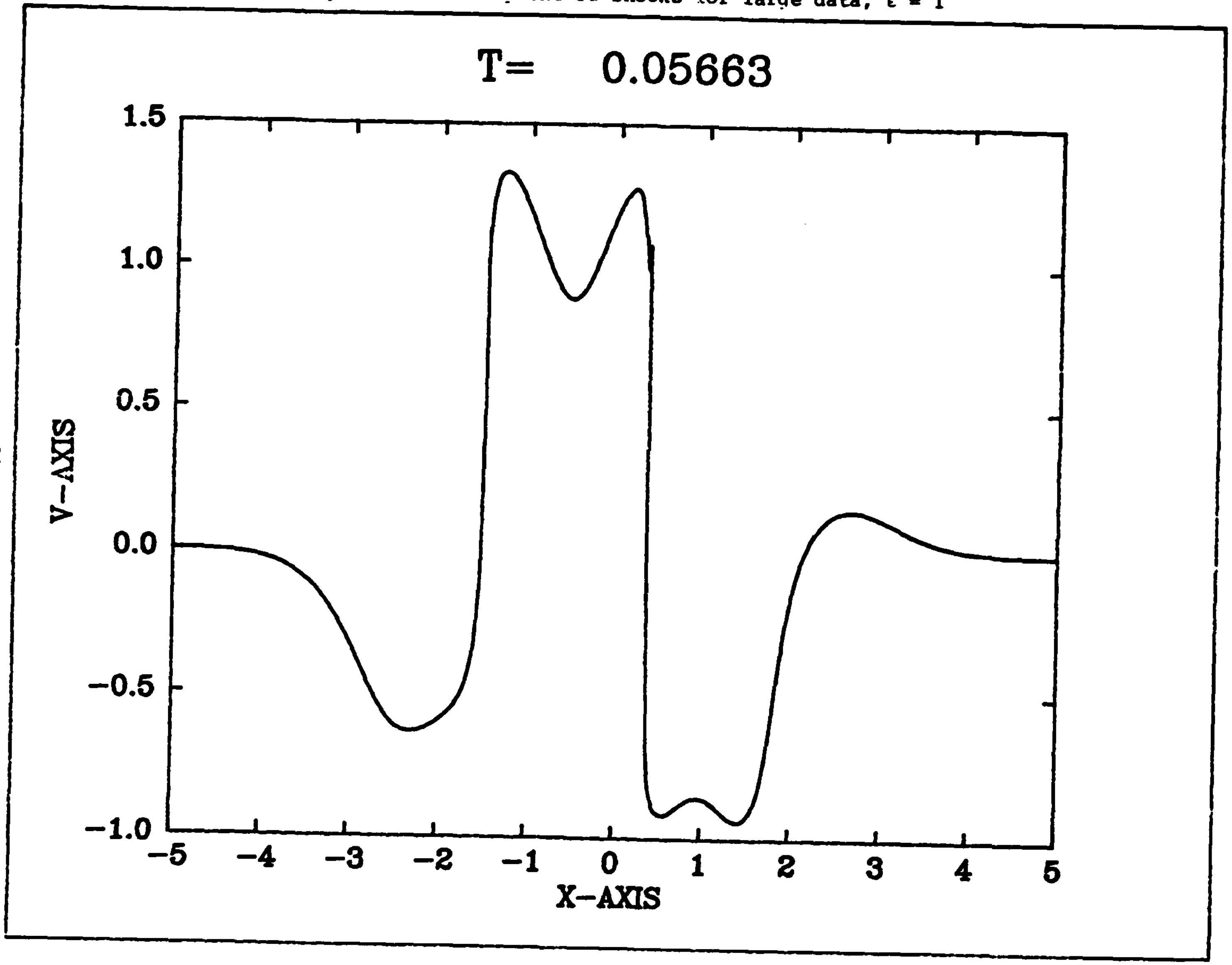


Figure 4. Development of Shocks for large data, ε = 1

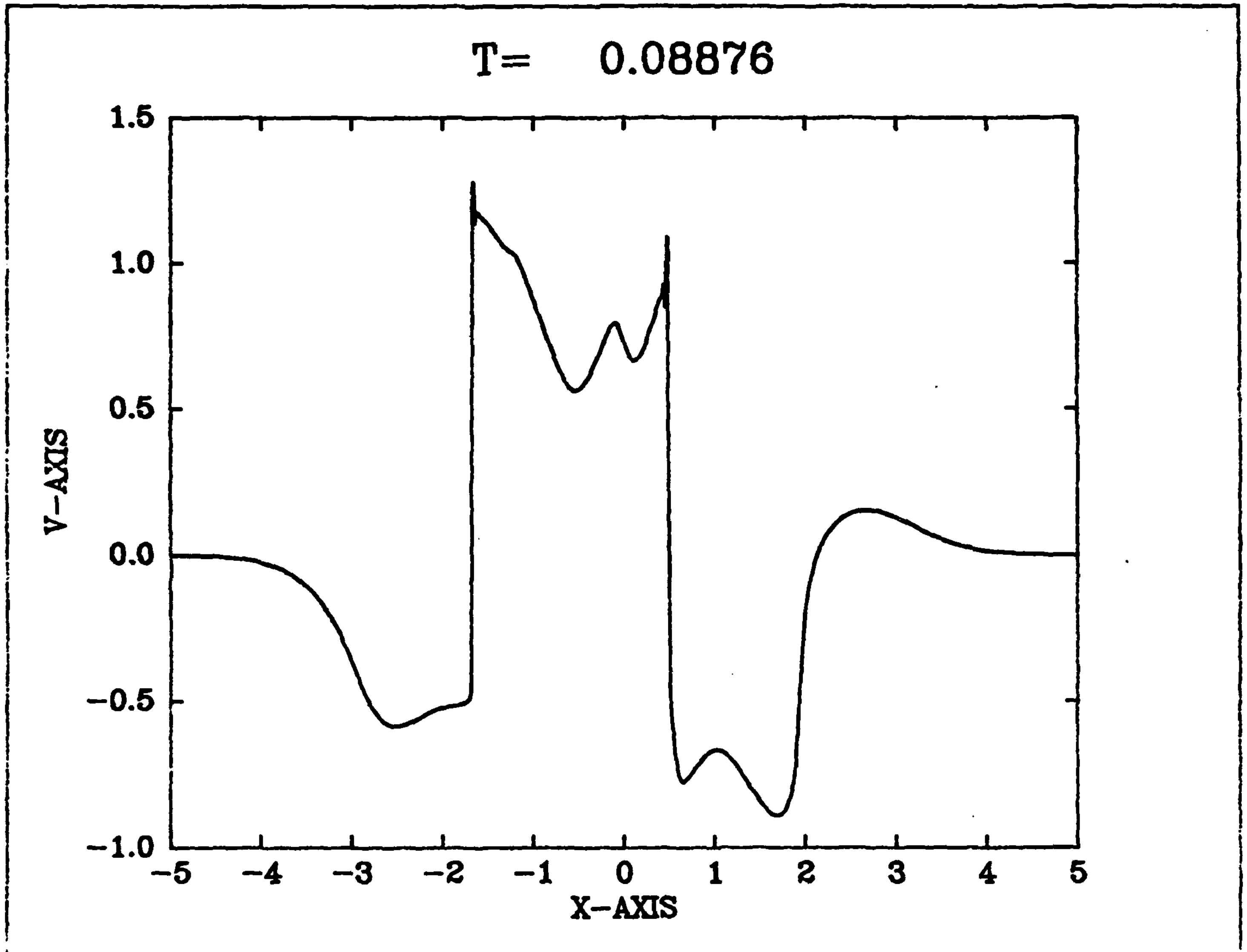


Figure 5. Development of shocks for large data, $\varepsilon = 1$

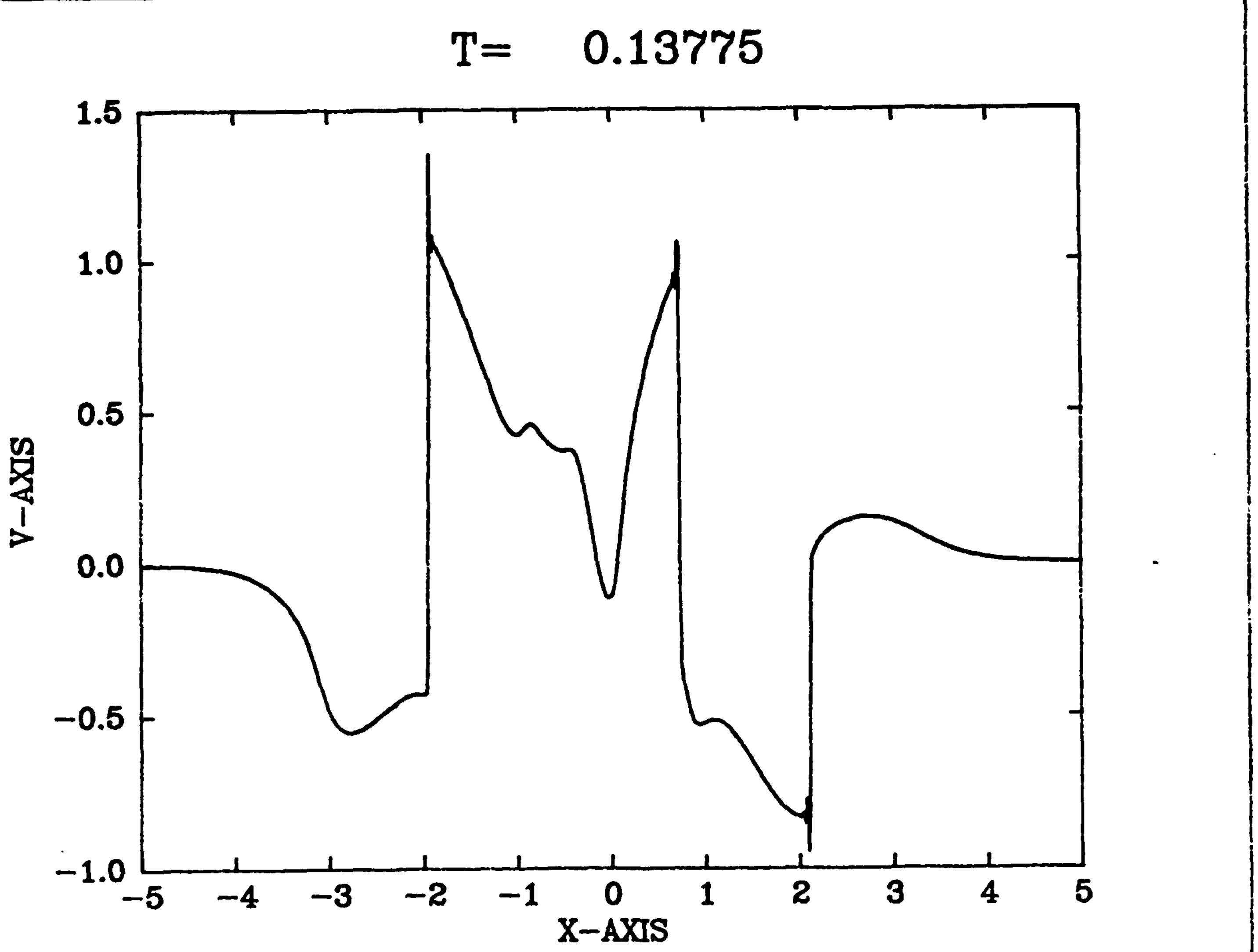


Figure 6. Development of shocks for large data, $\varepsilon = 1$

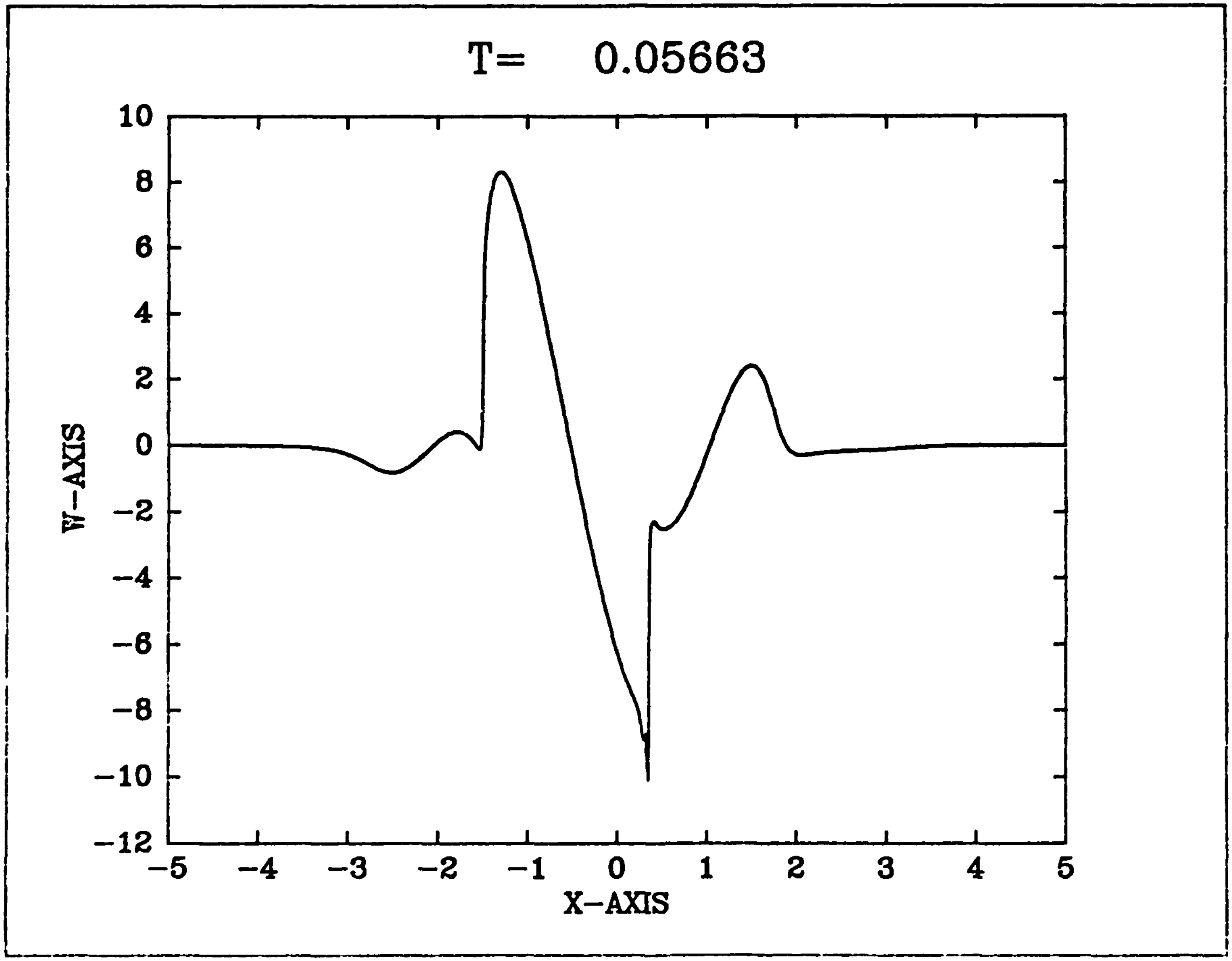


Figure 7. Development of shocks for large data, $\varepsilon = 1$

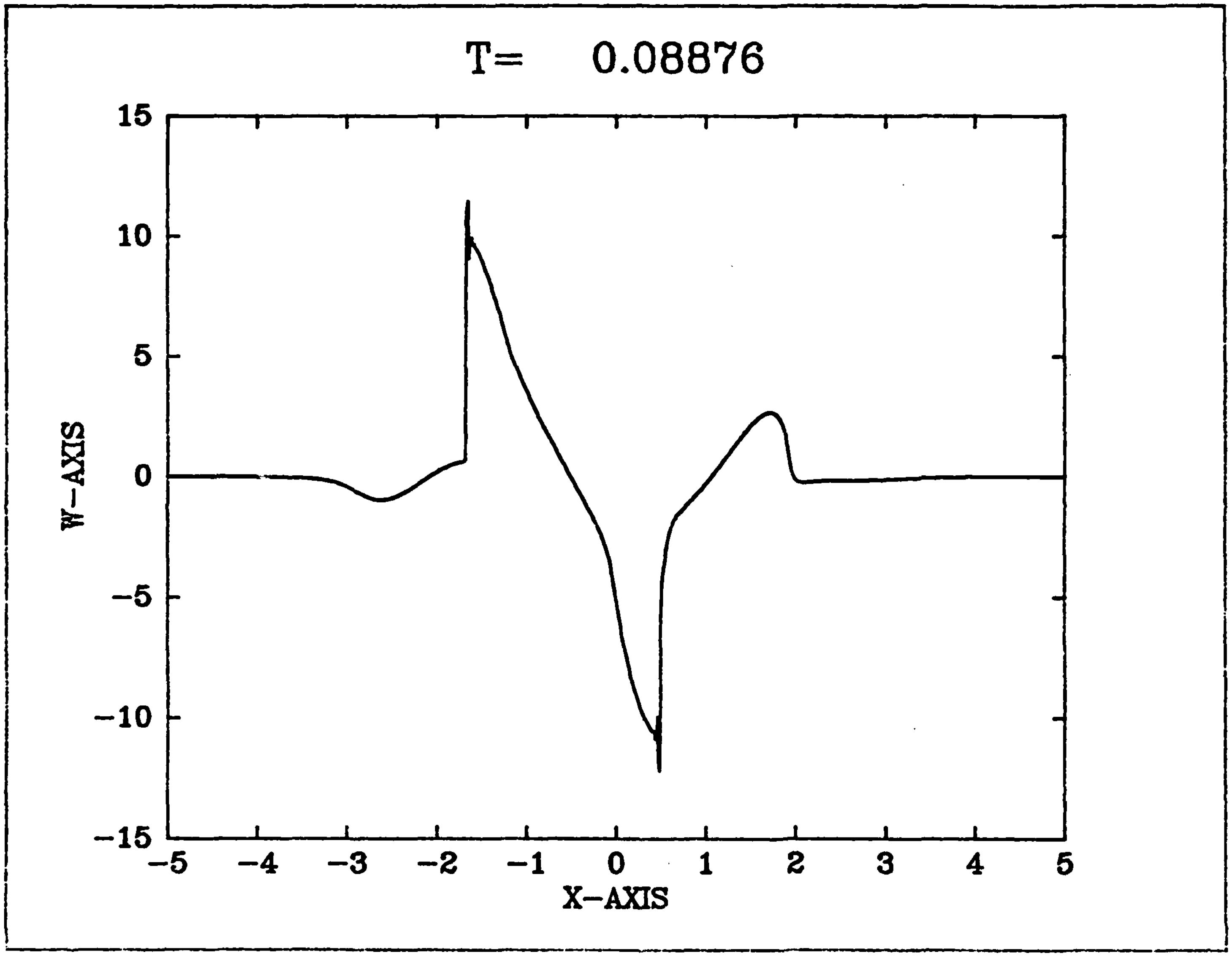


Figure 8. Development of shocks for large data, $\varepsilon = 1$

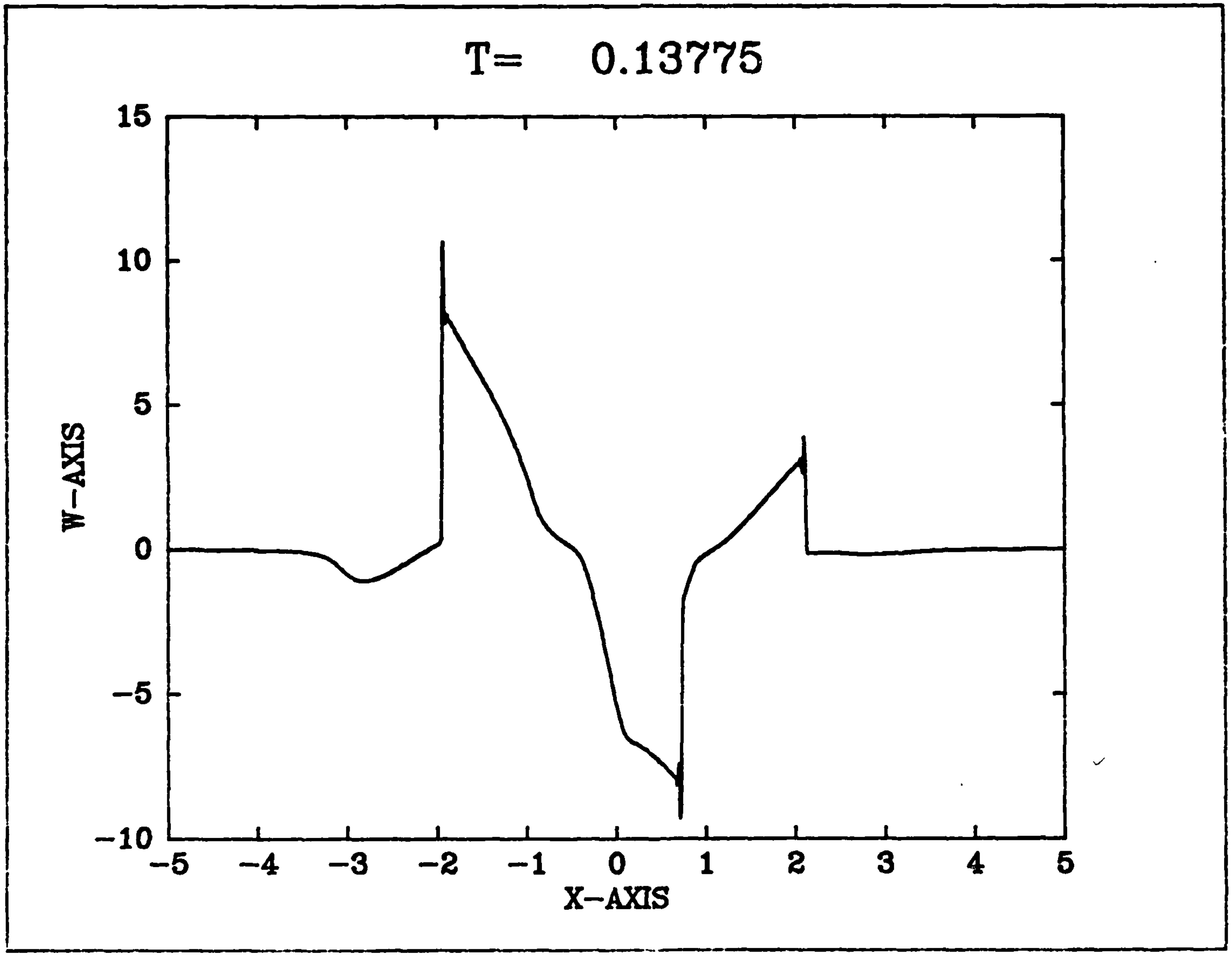


Figure 9. $V(X,t = 0.62099)$ for $\varepsilon = \frac{1}{5}$

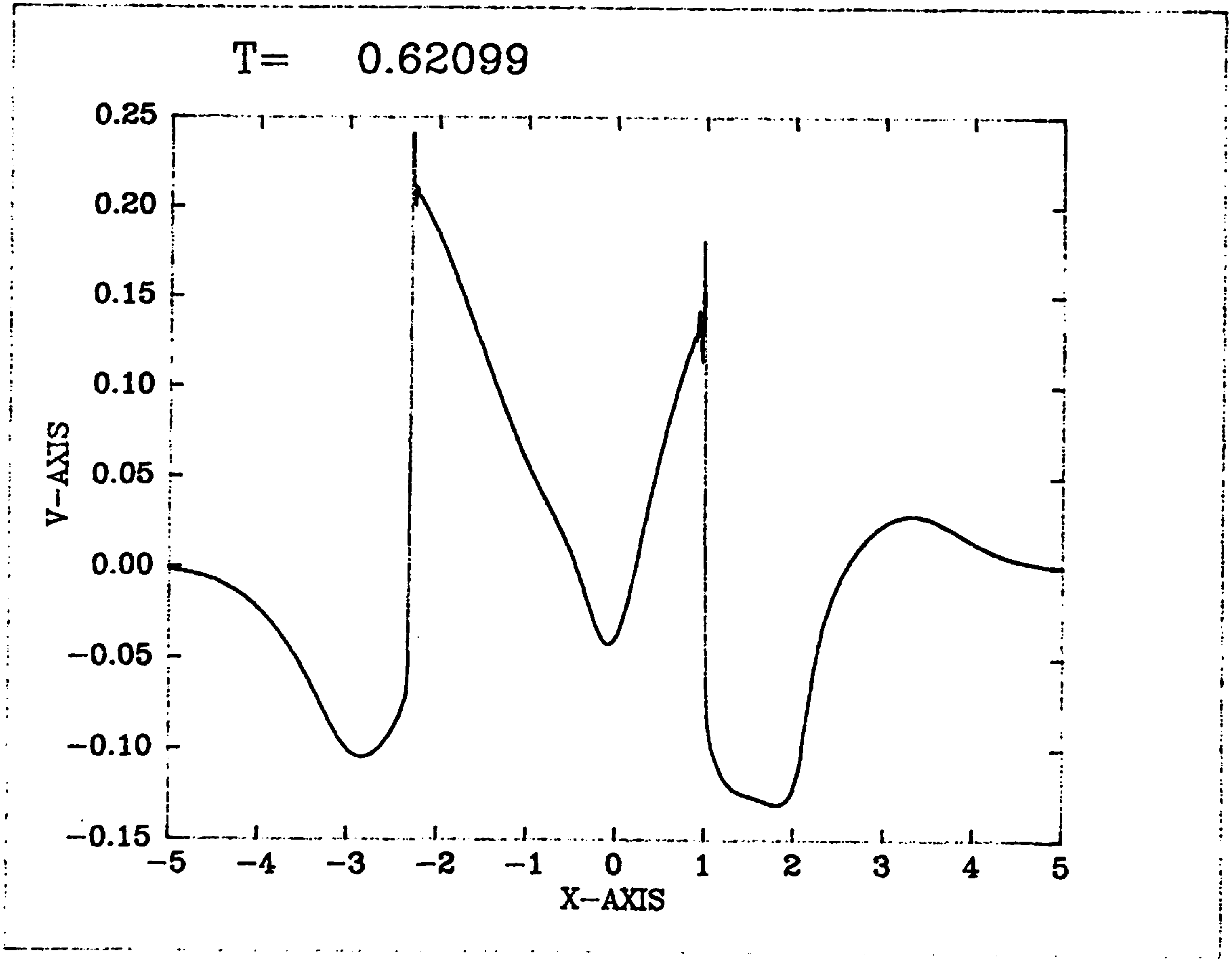


Figure 10. Dependence of shock width on h

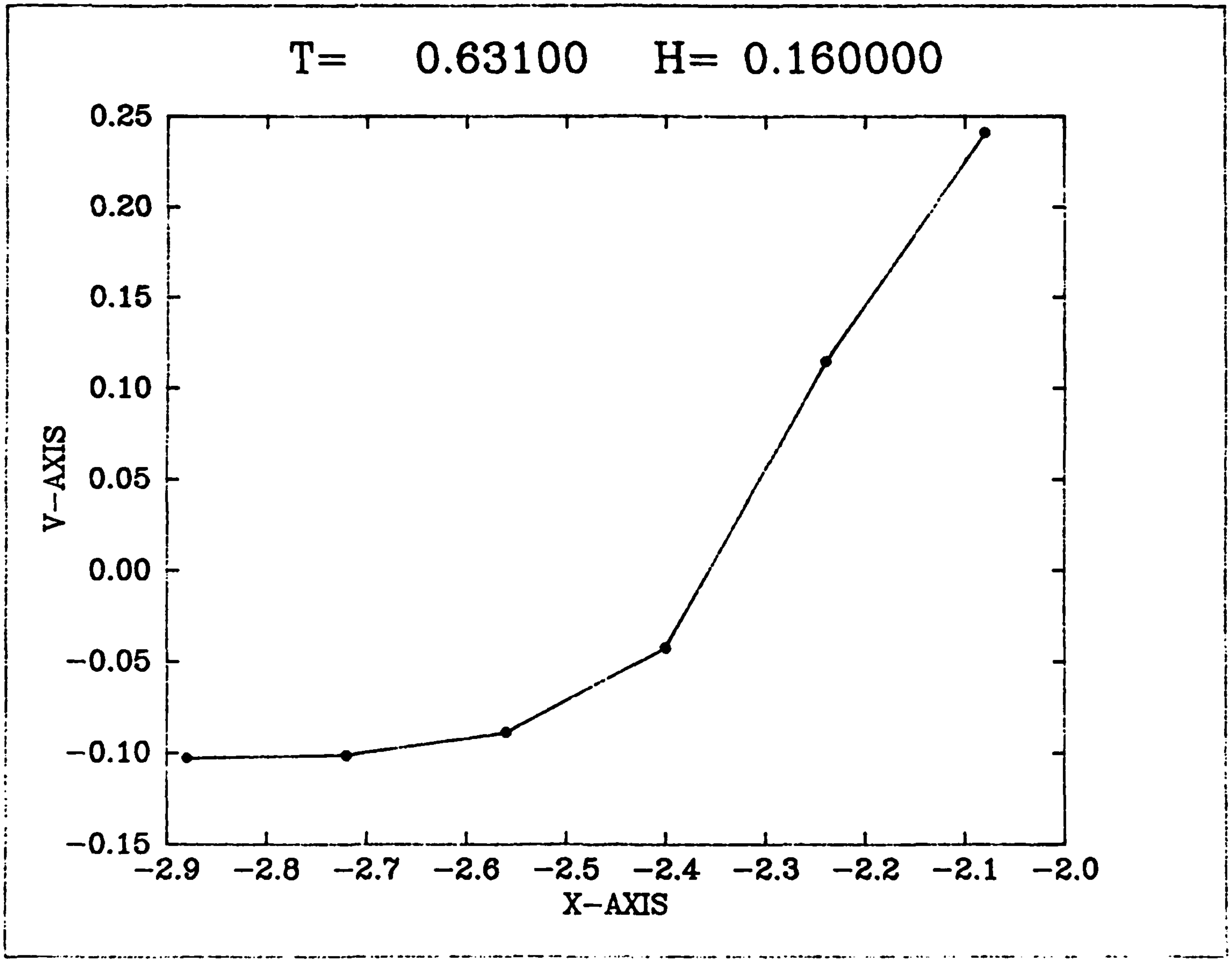


Figure 11. Dependence of shock width on h

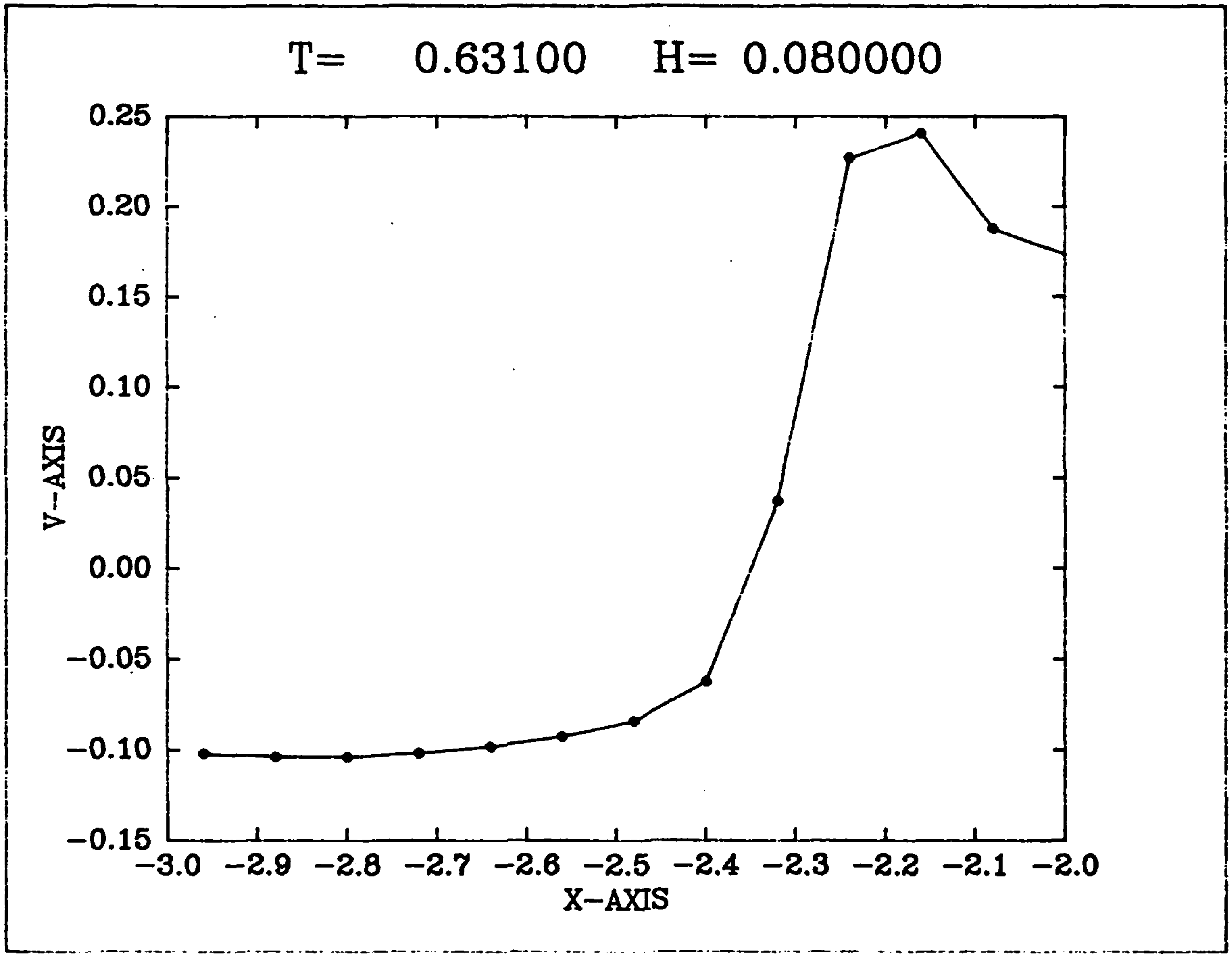


Figure 12. Dependence of shock width on h

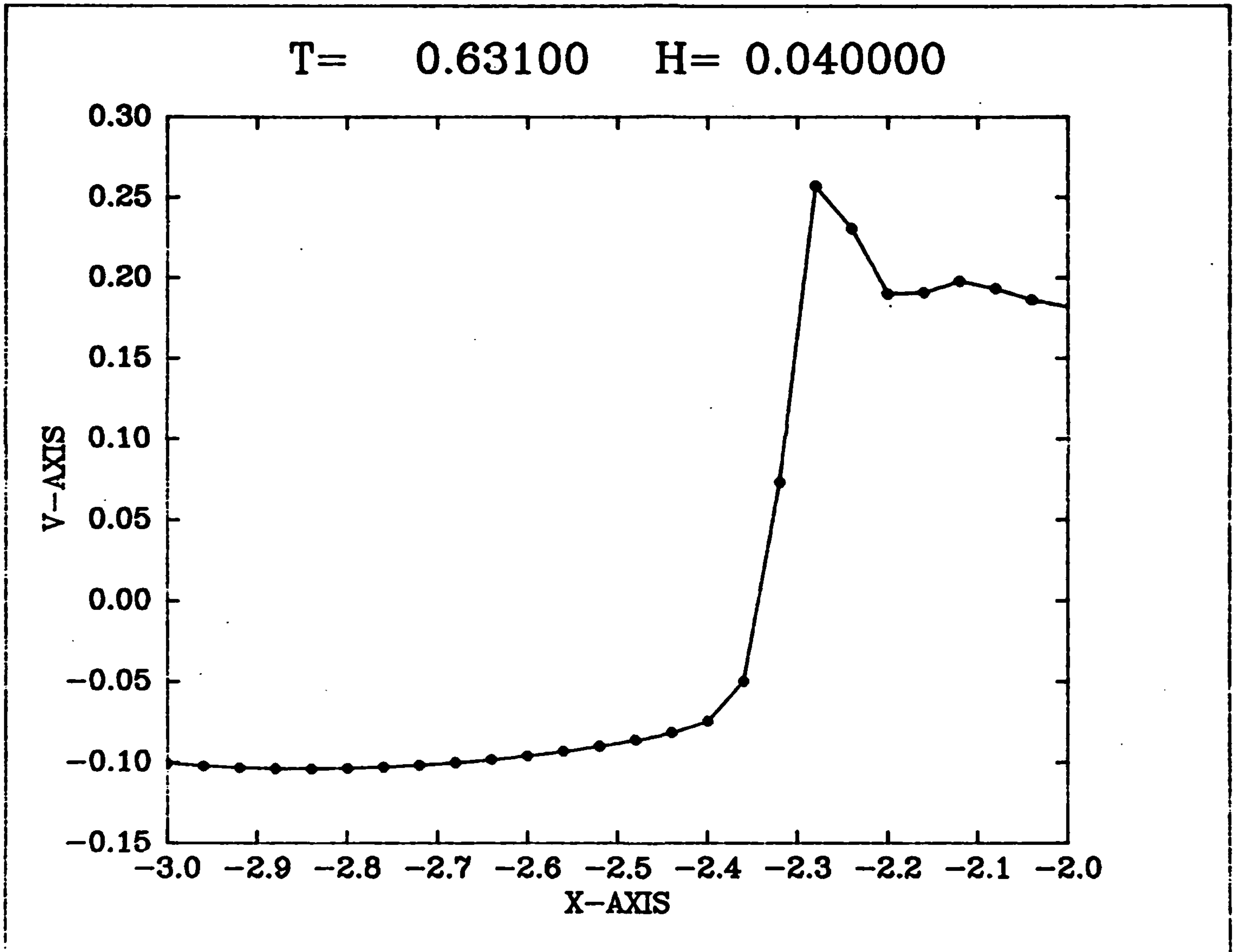


Figure 13. Dependence of shock width on h

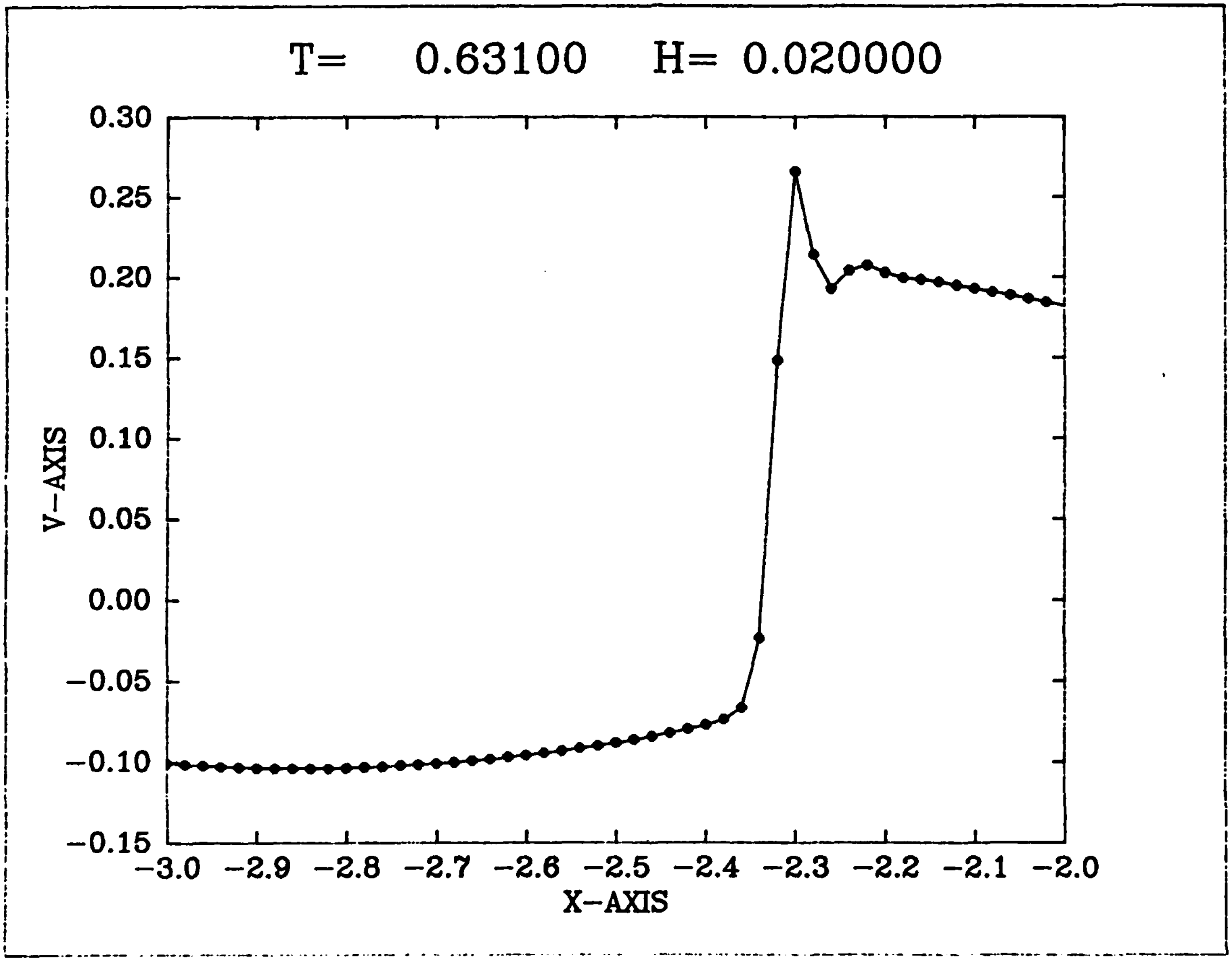


Figure 14. Dependence of shock width on h

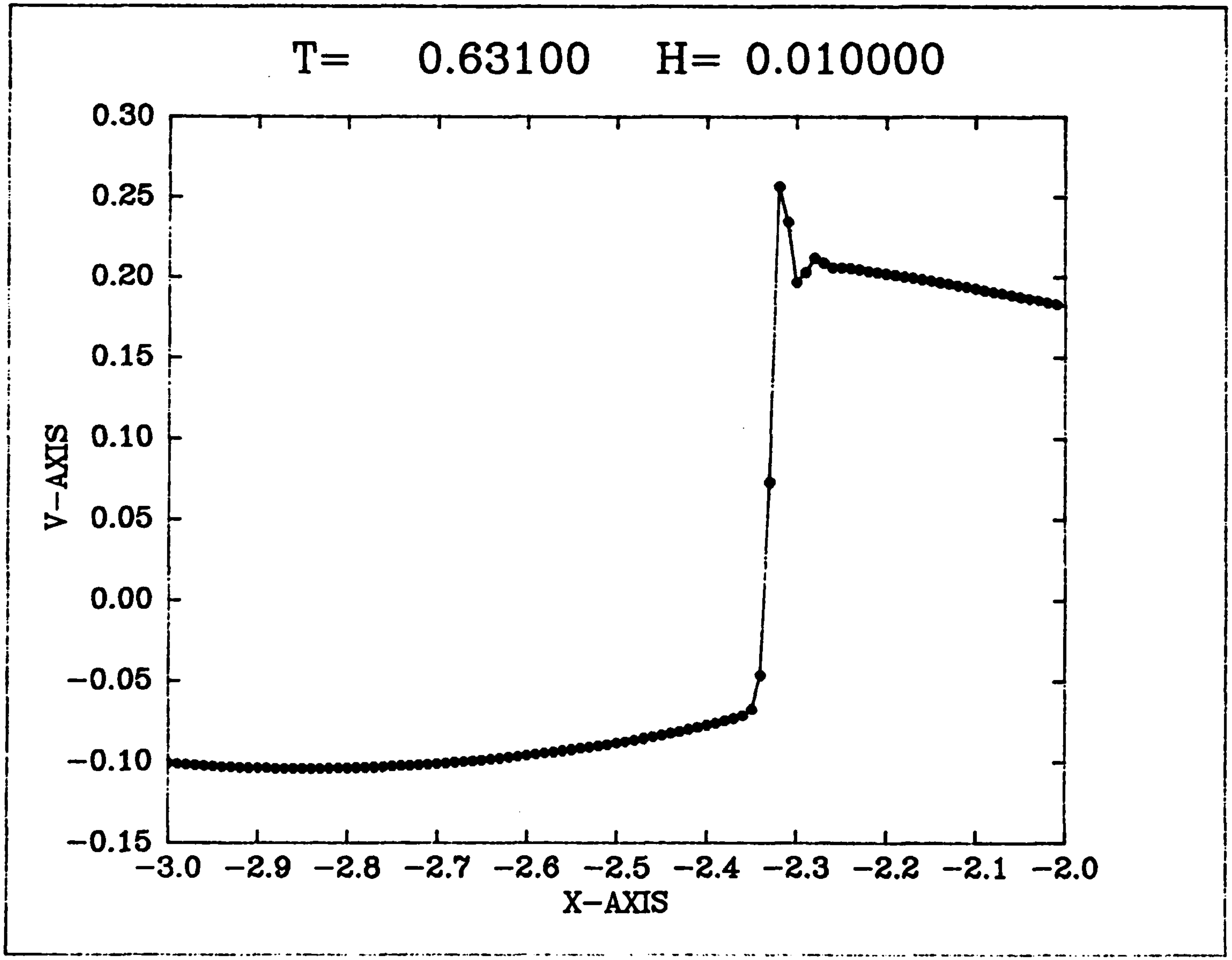


Figure 15. 'Small' Data, $\varepsilon = \frac{1}{40}$

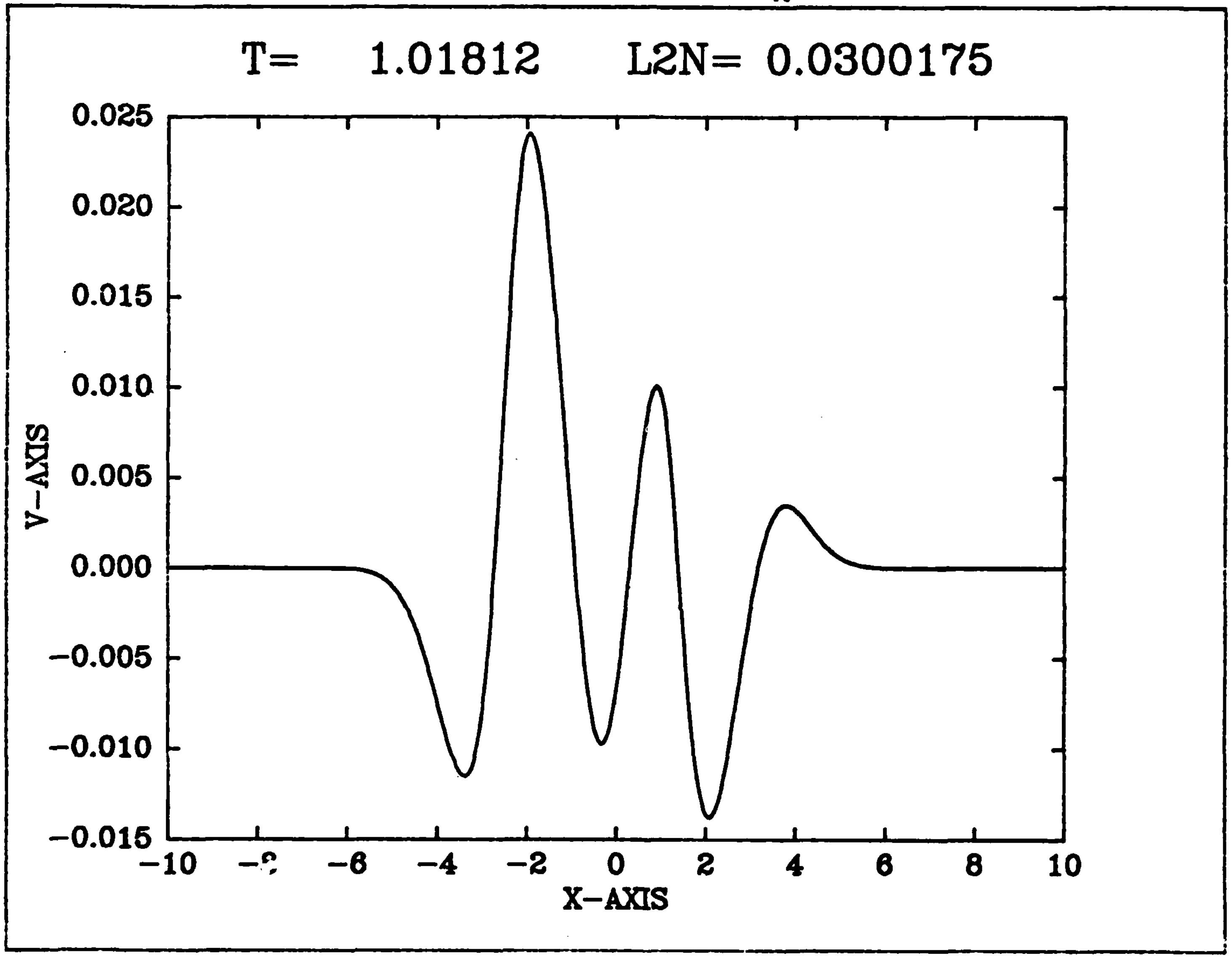


Figure 16. 'Small' Data, $\varepsilon = \frac{1}{40}$

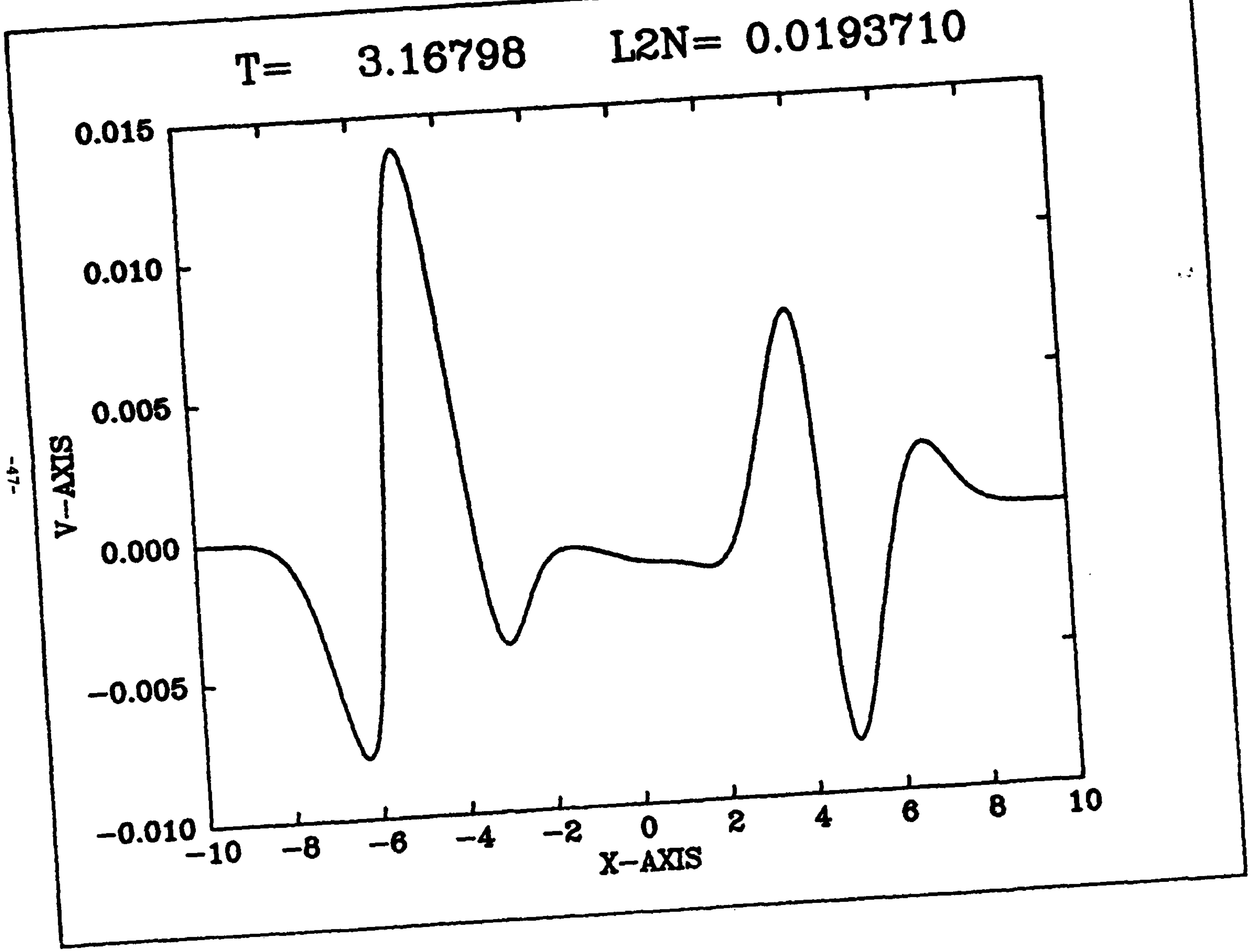


Figure 17. 'Small' Data, $\varepsilon = \frac{1}{40}$

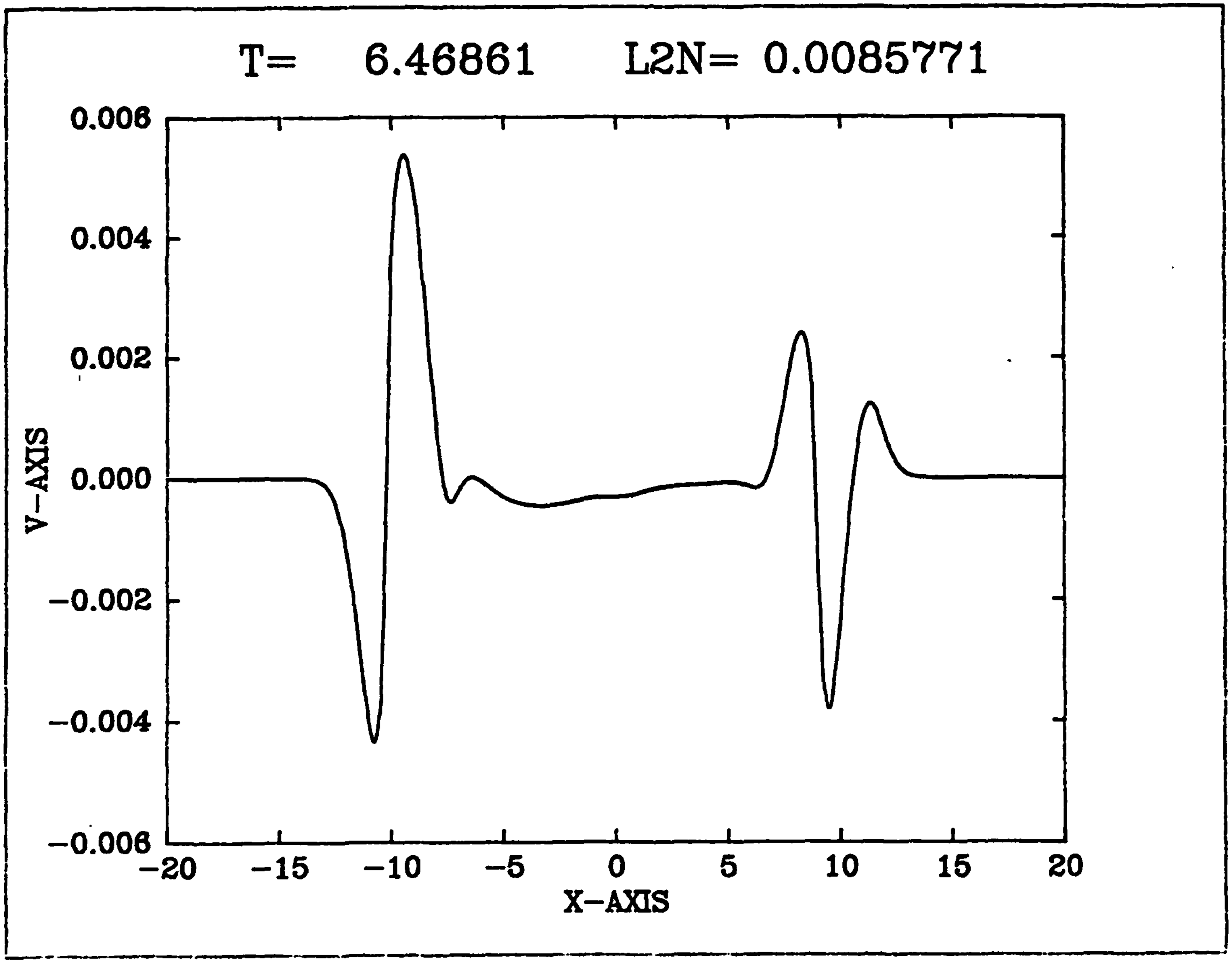


Figure 18. 'Small' Data, $\varepsilon = \frac{1}{40}$

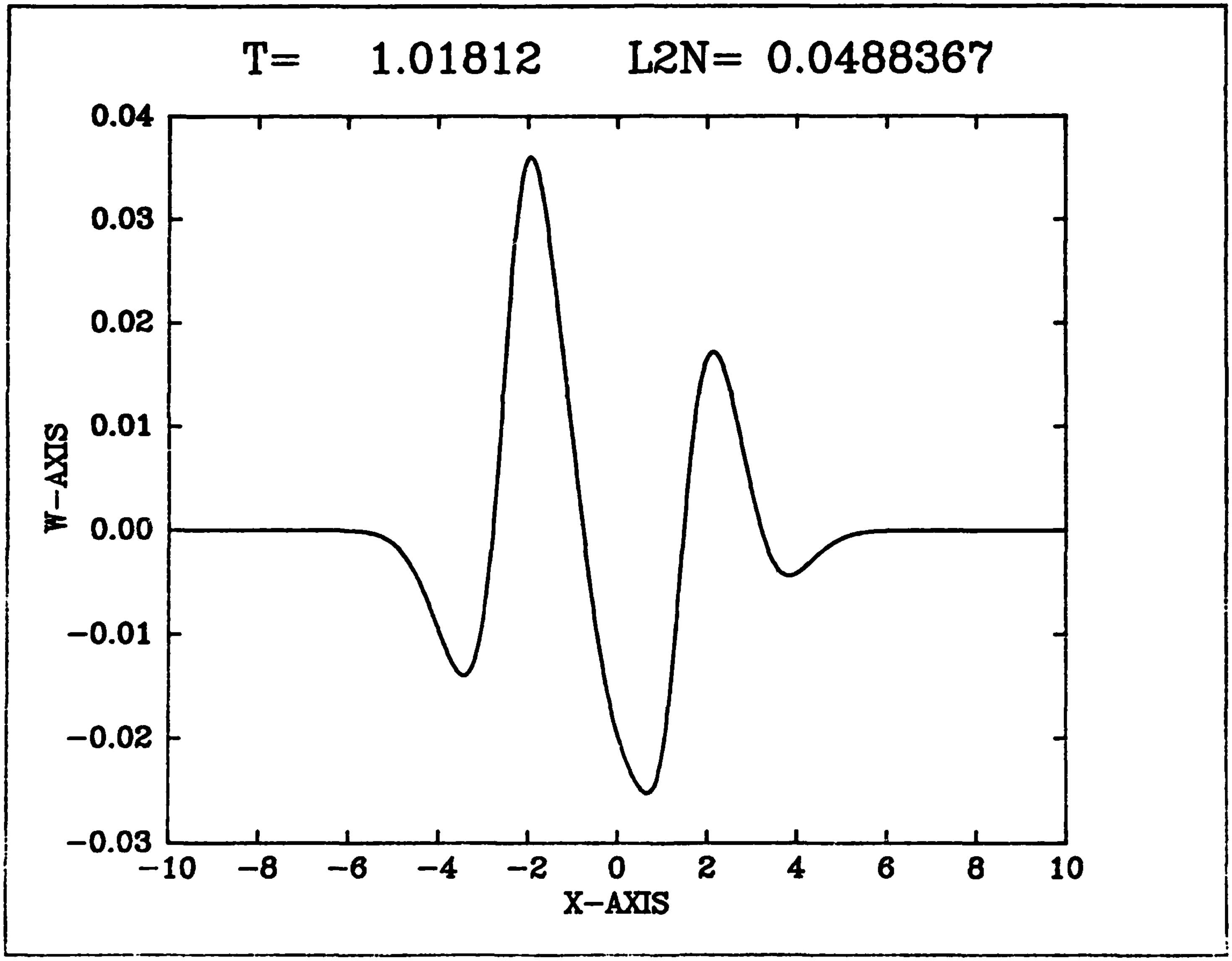


Figure 19. 'Small' Data, $\epsilon = \frac{1}{40}$

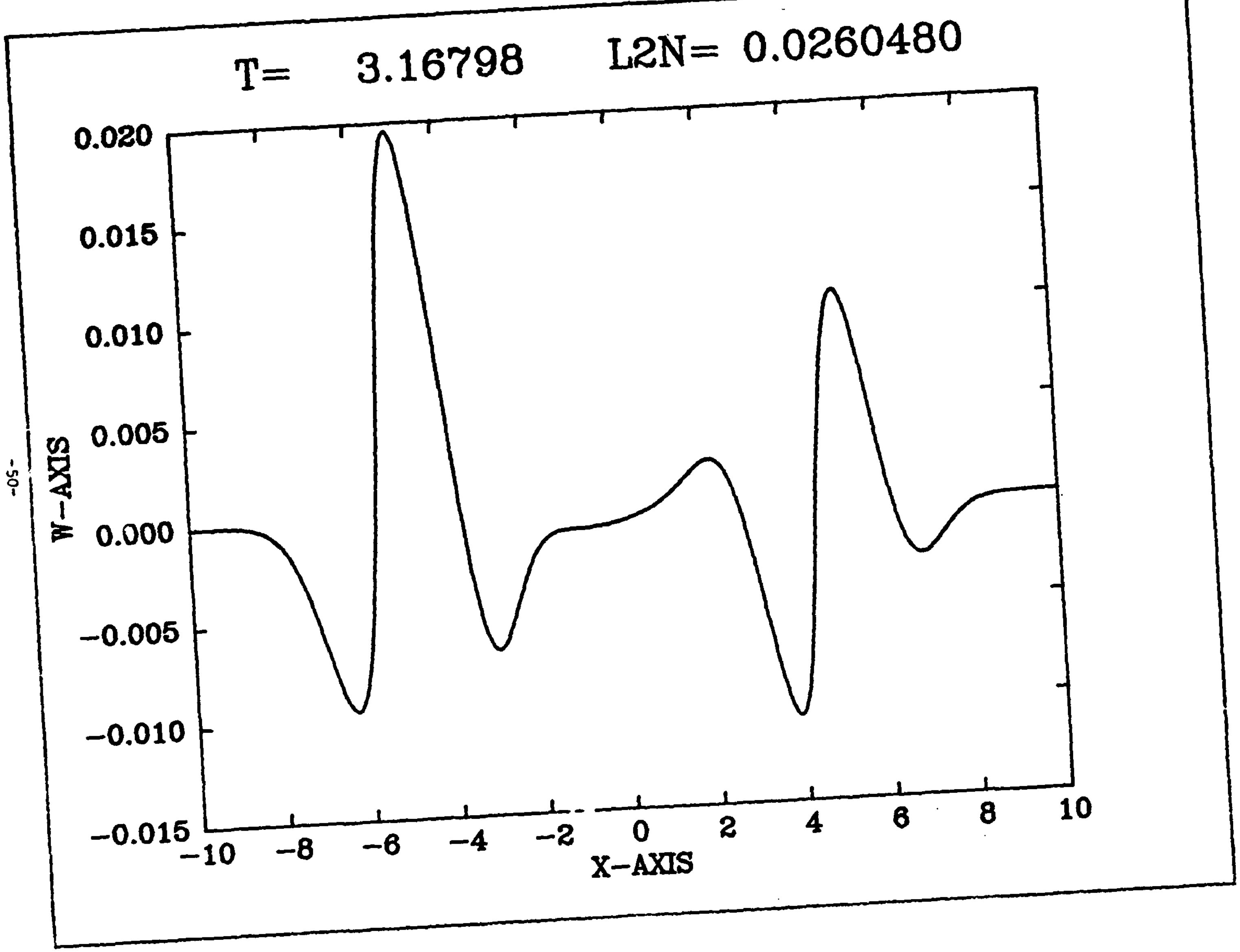


Figure 20. 'Small' Data, $\varepsilon = \frac{1}{40}$

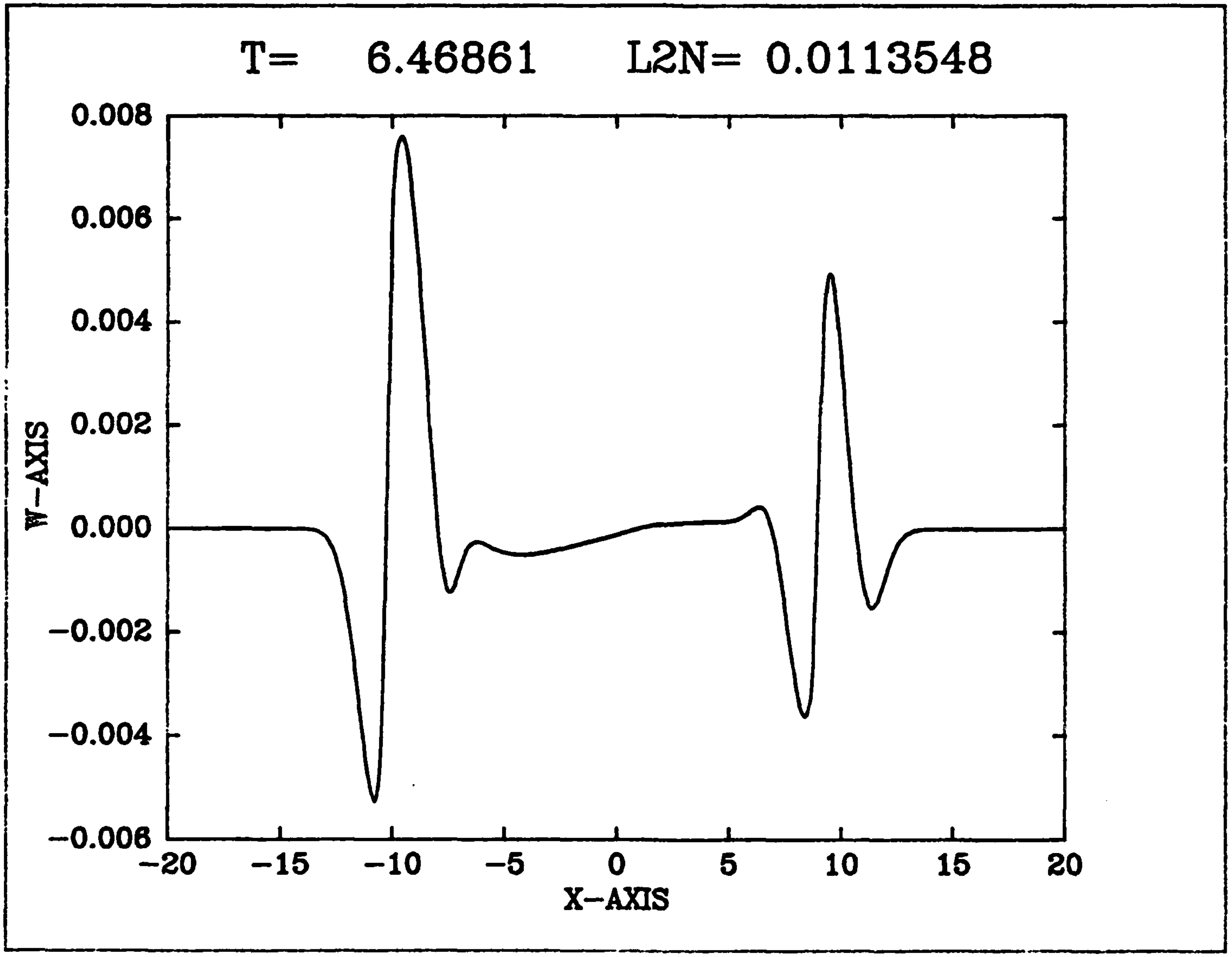


Figure 21. Wave Equation with $\varepsilon = \frac{1}{40}$

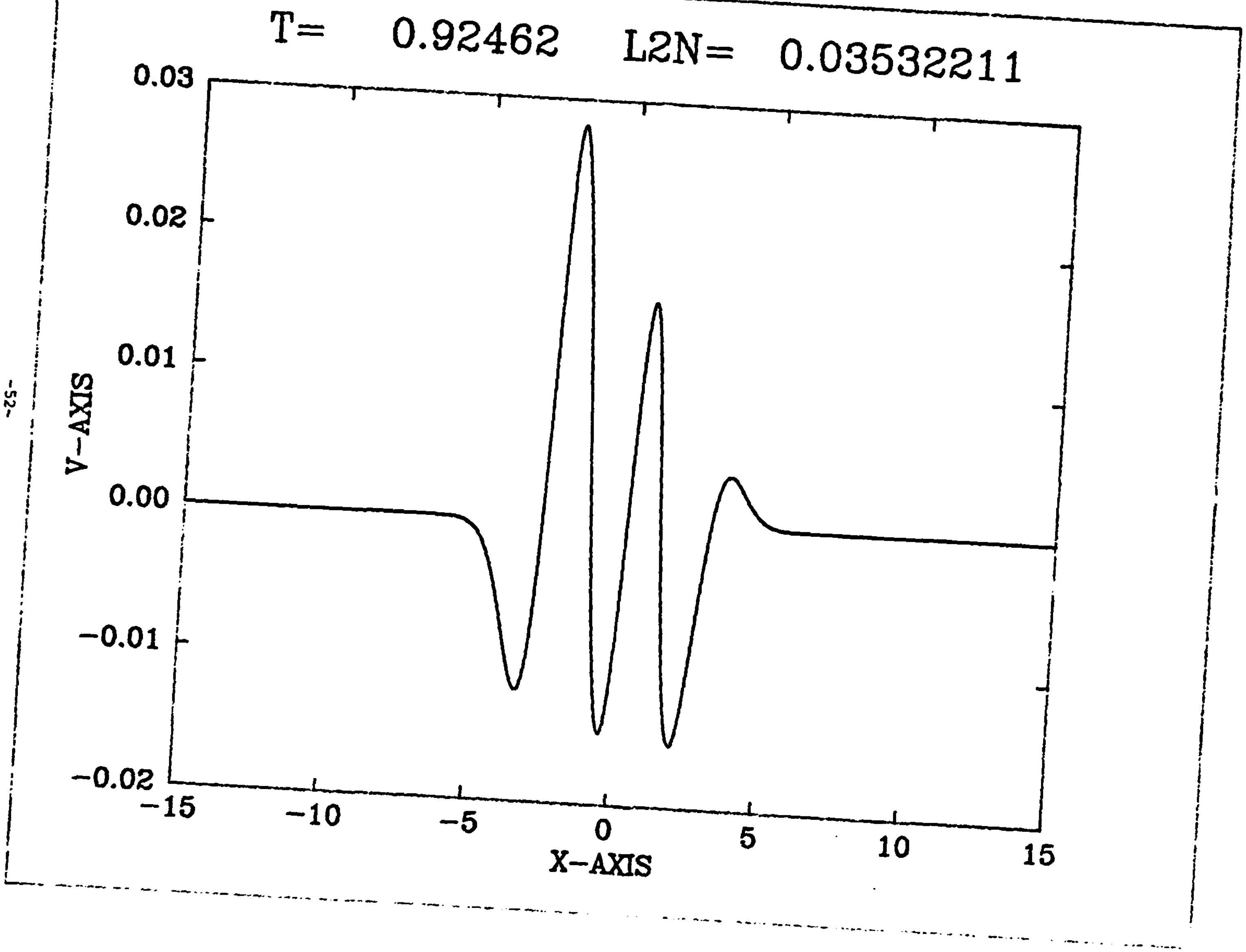


Figure 22. Wave Equation with $\epsilon = \frac{1}{40}$

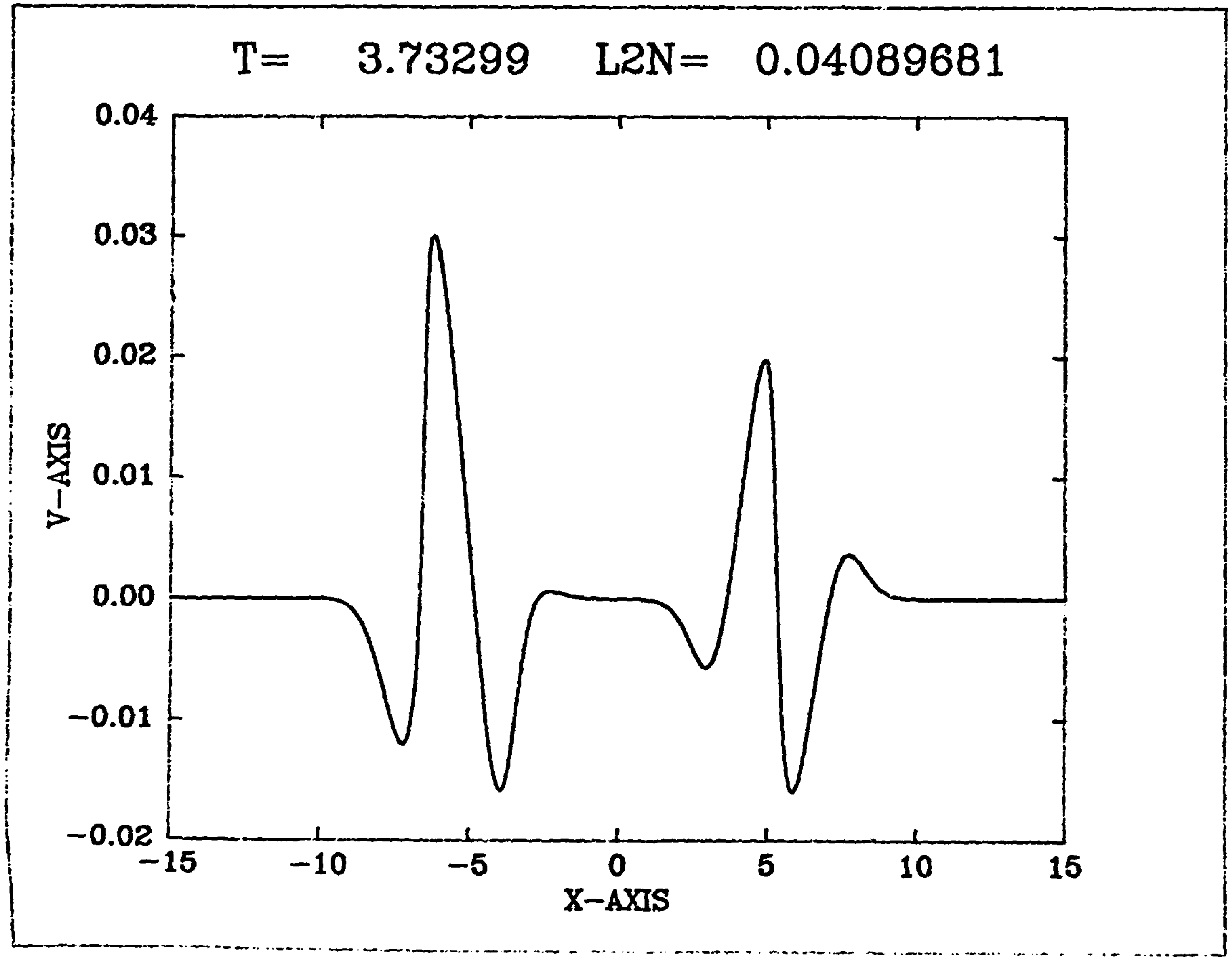


Figure 23. Wave Equation with $\varepsilon = \frac{1}{40}$

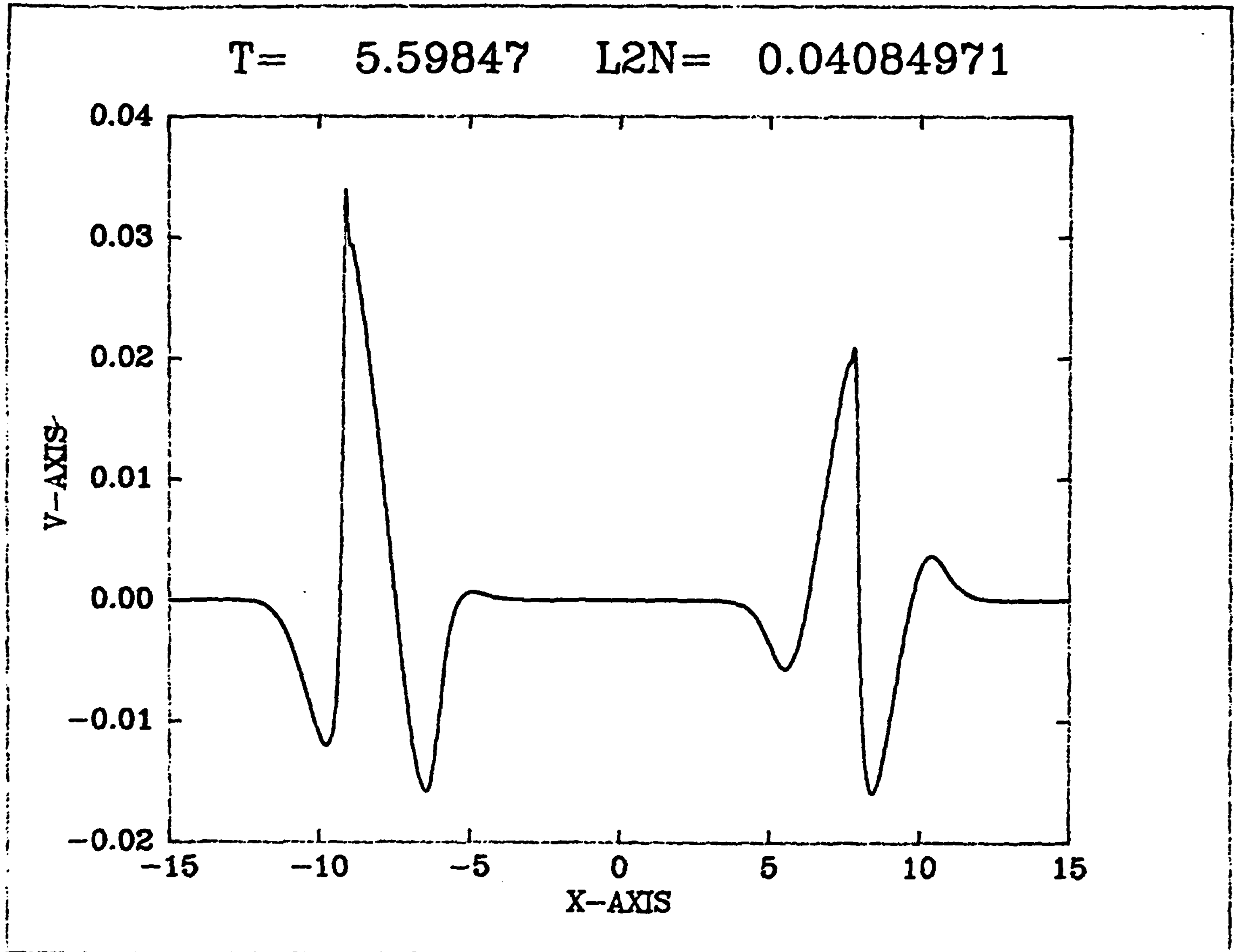


Figure 24. Wave Equation with $\varepsilon = \frac{1}{40}$

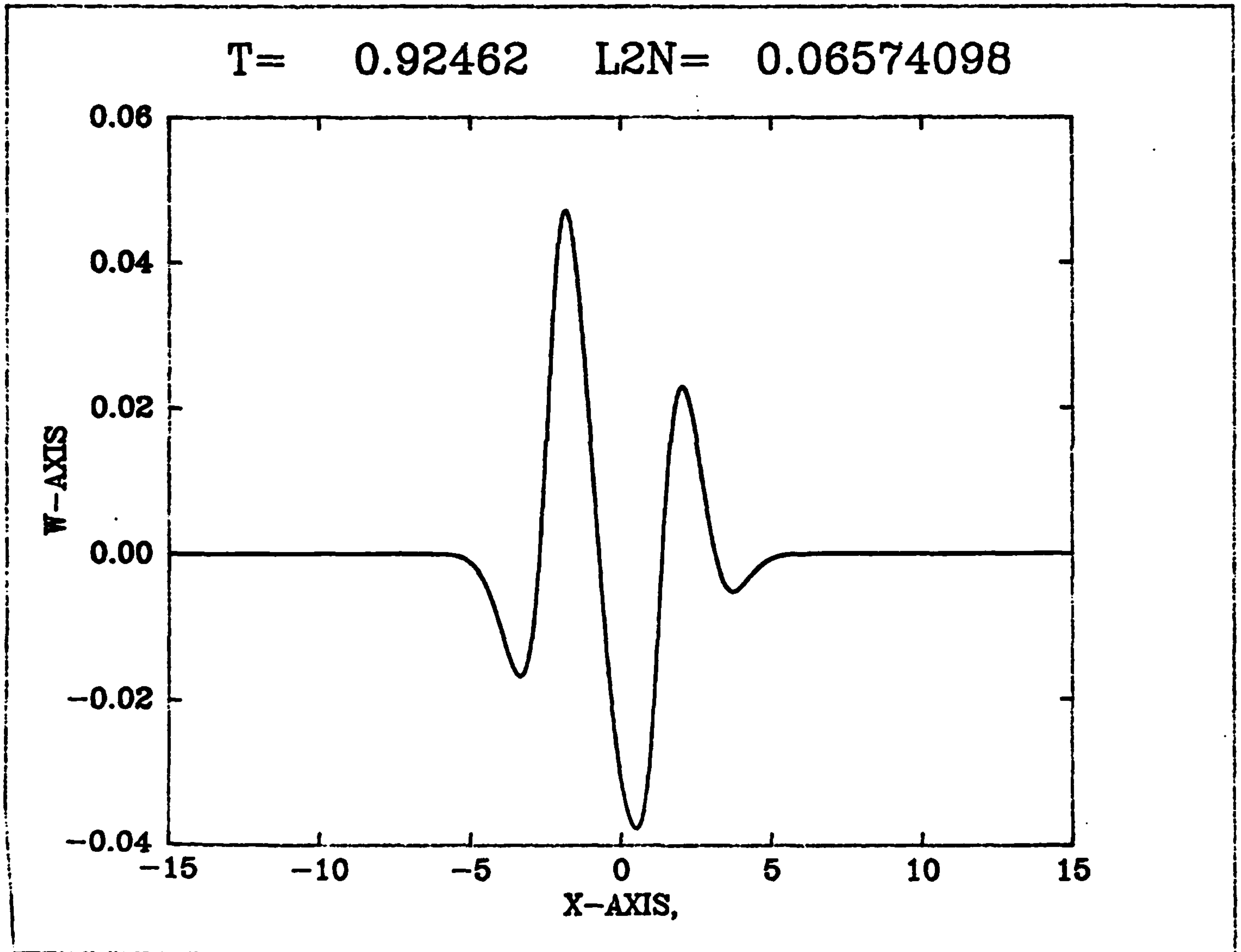


Figure 25. Wave Equation with $\varepsilon = \frac{1}{40}$

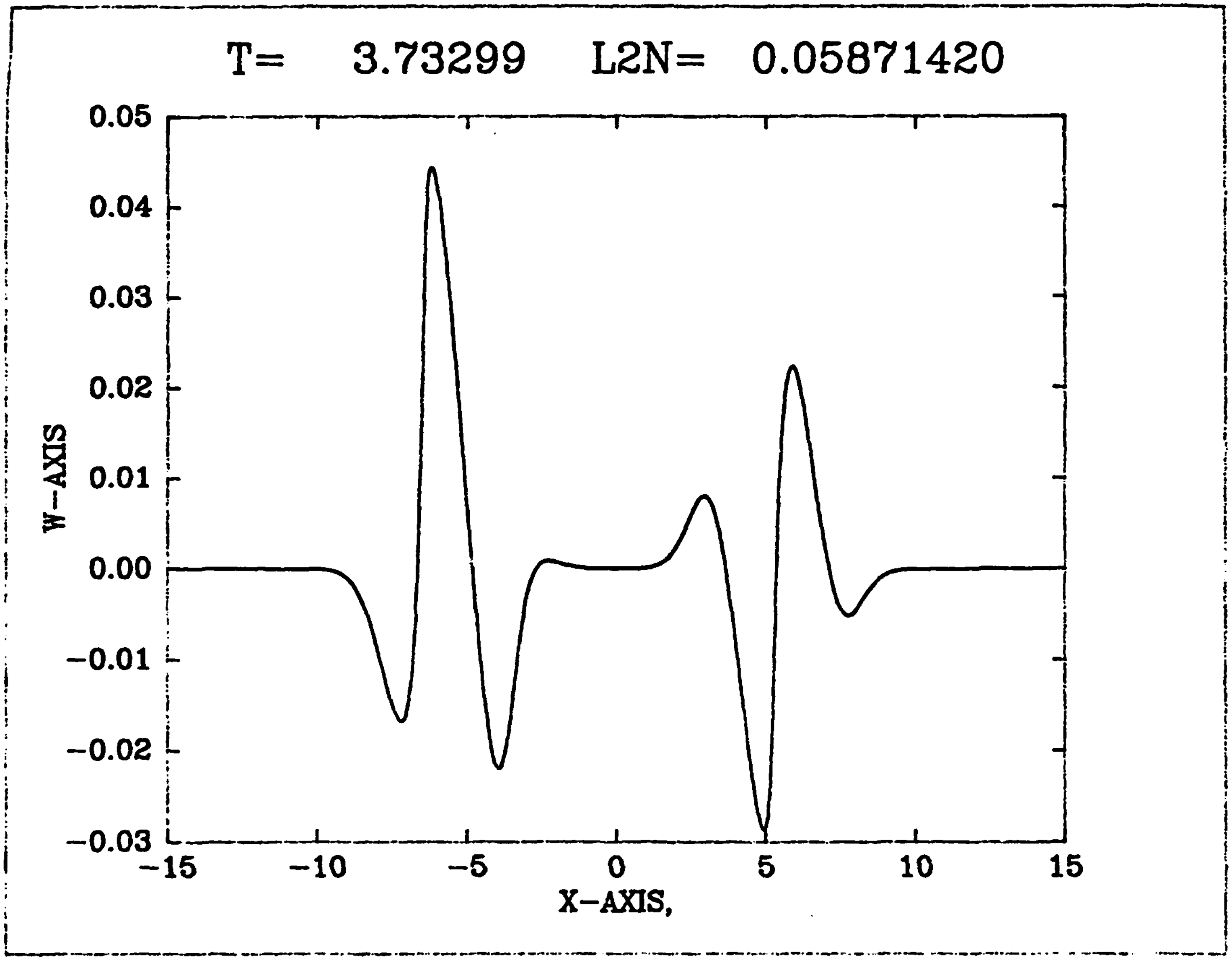
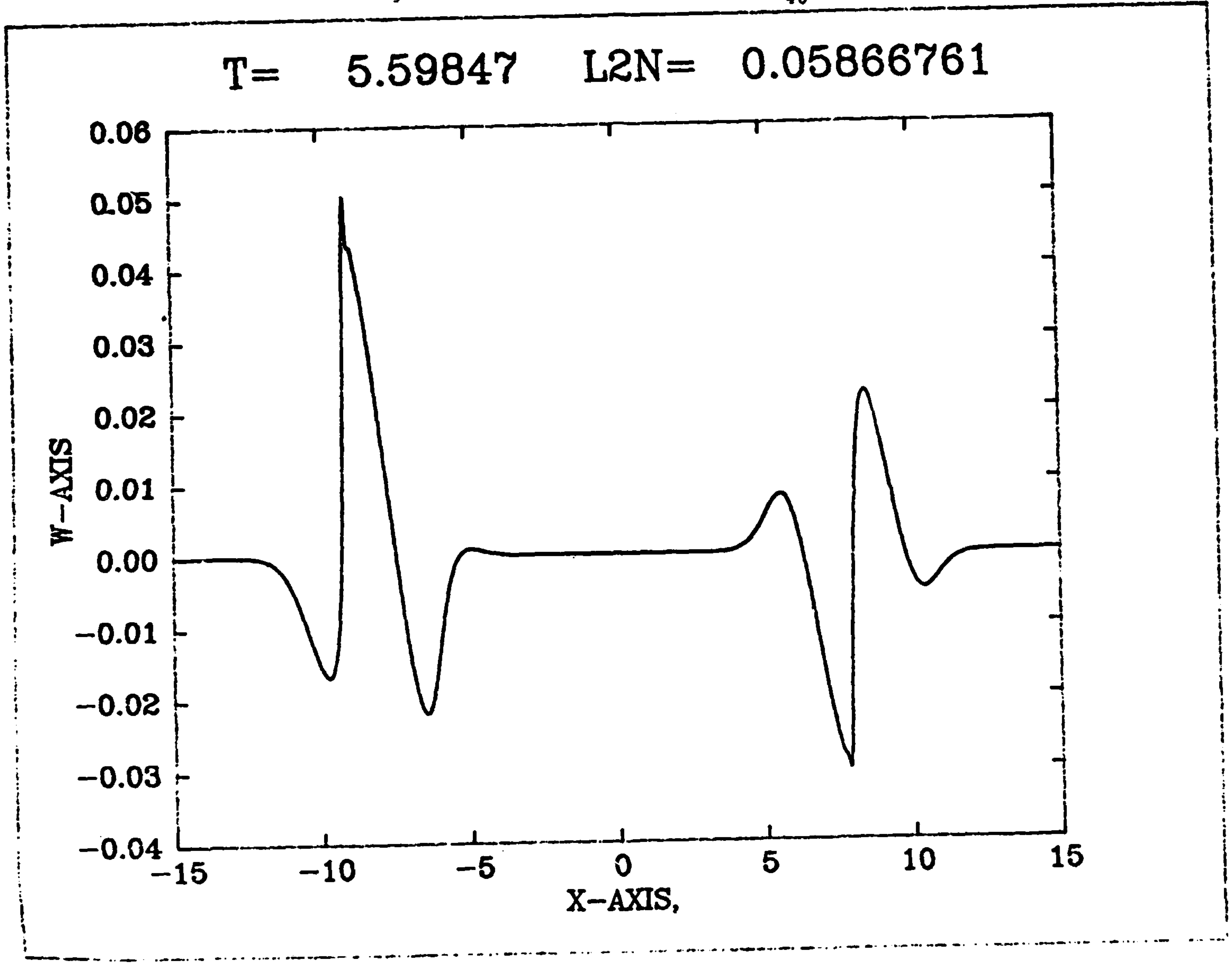


Figure 26. Wave Equation with $\epsilon = \frac{1}{40}$



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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper is concerned with Lax-Wendroff methods for a class of hyperbolic history value problems. These problems have the feature that globally (in time) smooth solutions exist if the data are sufficiently small and that solutions develop singularities for large data. We prove (second order) convergence of the Lax-Wendroff method for smooth solutions and investigate numerically the dependence on the initial data. We demonstrate the occurrence of shock type singularities and compare the results to the quasilinear wave equation (without Volterra term).

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